Fifteenth Int. Conf. on IFSs, Burgas, 11-12 May 2011
NIFS 17 (2011), 2, 35-43

On α- and α^*- separation axioms in intuitionistic fuzzy topological spaces

Rekha Srivastava and Amit Kumar Singh

Department of Applied Mathematics,Institute of Technology
Banaras Hindu University,Varanasi, 221005, India
E-mails: rekhasri@bhu.ac.in, amitkitbhu@gmail.com

Abstract: In this paper we introduce the gradation of the separation axioms T_0, T_1 and T_2 in an intuitionistic fuzzy topological space in the sense of Mondal and Samanta [6]. Using these concepts we have defined α- and α^*- T_i intuitionistic fuzzy topological spaces, $i = 0, 1, 2$, and studied them in detail.
Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy topological space, Separation axioms, Bifuzzy topological space.
AMS Classification: 54A40, 03E72

1 Introduction

In [1], Atanassov introduced intuitionistic fuzzy sets in a set X. In [6], Mondal and Samanta gave the concept of intuitionistic gradation of openness of fuzzy sets in X and using this, they defined an intuitionistic fuzzy topological space (IFTS, in short). Yue and Fang, in [15] considered the separation axioms T_0, T_1 and T_2 in an I-fuzzy topological space in the sense of Šostak [8] and Kubiak [4]. We extend and study these separation axioms in an intuitionistic fuzzy topological space in the sense of Mondal and Samanta [6]. In addition, we also define α- and α^*- separation axioms in this setting.

It is observed that if an IFTS (X, τ, τ^*) is α-T_i or α^*-T_i, then $T_i(X, \tau, \tau^*) \geq \alpha$ where $T_i(X, \tau, \tau^*)$ denotes the degree to which (X, τ, τ^*) is T_i, $i = 0, 1, 2$. Further it is proved that all these separation properties satisfy the hereditary, productive and projective properties.

2 Preliminaries

Let X be a nonempty set. By I^X, where $I = [0, 1]$, we denote the set of all fuzzy sets in X i.e. all functions from X to I. For a fuzzy set $A \in I^X$ A' will denotes its (Zadeh [16]) complement. For $\alpha \in I$, $\underline{\alpha}$ will denote the α-valued constant fuzzy set in X. Each $Y \subseteq X$ will be identified with the fuzzy set in X which is its I-valued characteristic function, which too will be denoted as Y.

Definition 2.1. (Wong [14]). A fuzzy point x_r in X is a fuzzy set in X taking value $r \in (0, 1)$ at x and zero elsewhere. A fuzzy singleton (Zadeh [17]) x_r in X is a fuzzy set in X taking value $r \in (0, 1]$. Here x and r are respectively called the support and value of x_r.
A fuzzy point x_r is said to belong to a fuzzy set A if $r < A(x)$. It can be easily seen that $x_r \in \cup_{i \in \Lambda} A_i \Leftrightarrow x_r \in A_i$ for some $i \in \Lambda$.
Two fuzzy points/fuzzy singletons are said to be distinct if their supports are distinct.

Definition 2.2. (Pu and Liu [5]). Let x_r be a fuzzy singleton in X and $A \in I^X$. Then x_r is said to be quasi-coincident with A (notation: $x_r q A$) if $A(x) + r > 1$. Two fuzzy sets A, B in X are said to be quasi-coincident (notation: AqB) if $A(x) + B(x) > 1$ for some $x \in X$. The relation (is not quasi-coincident) is denoted by $\neg q$.

We use the well known notion of 'fuzzy topology'as given in Chang [2]

Definition 2.3. (Pu and Liu [5]). Let (X, τ) be a fuzzy topological space in the sense of Chang and x_r be a fuzzy singleton. Then a Q-neighborhood (in short, Q-nbd) of a fuzzy singleton x_r is a fuzzy set $N \in I^X$ such that there exists $U \in \tau$ with $x_r q U \subseteq N$.

Definition 2.4. (Sostak [8], Kubiak [4]). An I- fuzzy topology on a set X is a map $\tau : I^X \longrightarrow I$ such that

(i) $\tau(\underline{1}) = \tau(\underline{0}) = 1$;

(ii) $\tau(U \cap V) \geq \tau(U) \wedge \tau(V), \forall U, V \in I^X$;

(iii) $\tau(\bigcup_{i\in\Lambda} U_i) \geq \bigwedge_{i\in\Lambda} \tau(U_i), \forall U_i \in I^X, i \in \Lambda$.

The pair (X, τ) is called an I- fuzzy topological space.

Definition 2.5. (Mondal and Samanta [6]). Let X be a nonempty set. An intuitionistic gradation of openness (in short, IGO) of fuzzy sets of X is an ordered pair (τ, τ^*) of functions from I^X to I such that

(i) $\tau(U) + \tau^*(U) \leq 1, \forall U \in I^X$;

(ii) $\tau(\underline{0}) = \tau(\underline{1}) = 1, \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$;

(iii) $\tau(U_1 \cap U_2) \geq \tau(U_1) \wedge \tau(U_2)$ and $\tau^*(U_1 \cap U_2) \leq \tau^*(U_1) \vee \tau^*(U_2), U_i \in I^X, i = 1, 2$;

(iv) $\tau(\bigcup_{i\in\Lambda} U_i) \geq \bigwedge_{i\in\Lambda} \tau(U_i)$ and $\tau^*(\bigcup_{i\in\Lambda} U_i) \leq \bigvee_{i\in\Lambda} \tau(U_i), U_i \in I^X, i \in \Lambda$.

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (IFTS, in short), where τ and τ^* may be interpreted as gradation of openness and gradation of non openness respectively.

Proposition 2.1. (Mondal and Samanta [6]). Let (X, τ, τ^*) be an IFTS. Then,

$$\tau_r = \tau^{-1}[r, 1] \text{ and } \tau^*_r = (\tau^*)^{-1}[0, 1 - r], r \in I_0$$

are two descending families of fuzzy topologies on X such that $\tau_r \subseteq \tau^*_r$.

Definition 2.6. (Mondal and Samanta [6]).

(1) Let (X, τ, τ^*) be an IFTS and $Y \subseteq X$. Then, the IFTS (Y, τ_Y, τ^*_Y) is called a subspace of (X, τ, τ^*) where $\tau_Y : I^Y \longrightarrow I$ and $\tau^*_Y : I^Y \longrightarrow I$ are defined as follows:
$\tau_Y(U) = \vee\{\tau(V) : V \in I^Y, V \mid Y = U\}$
and $\tau^*_Y(U) = \wedge\{\tau^*(V) : V \in I^Y, V \mid Y = U\}$.

(2) Let $\{(X_j, \tau_j, \tau^*_j) : j \in J\}$ be a family of IFTSs, $X = \Pi_{j\in J} X_j$ and $\{p_j : X \longrightarrow (X_j, \tau_j, \tau^*_j)\}_{j\in J}$ be the projection mappings. Then, the product IGO on X, denoted by $(\Pi_{j\in J}\tau_j, \Pi_{j\in J}\tau^*_j)$, which is defined as follows:
$(\Pi\tau_j)(U) = \vee\{r : U \in T_r\}$ and $(\Pi\tau^*_j)(U) = \wedge\{1 - r : U \in T^*_r\}$,
where T_r and T^*_r are fuzzy topologies on X, generated respectively by $\bigcup_{j\in J} T_{j,r}$ and $\bigcup_{j\in J} T^*_{j,r}$
where $T_{j,r} = \{p_j^{-1}(U) : U \in (\tau_j)_r\}$ and $T^*_{j,r} = \{p_j^{-1}(U) : U \in (\tau^*_j)_r\}$.
$(X, \Pi_{j\in J}\tau_j, \Pi_{j\in J}\tau^*_{ij})$ is called the product IFTS of the family $\{(X_j, \tau_j, \tau^*_j)\}_{j\in J}$.

(3) Let (X, τ, τ^*) and (Y, δ, δ^*) be two IFTSs and $f : X \longrightarrow Y$ be a mapping. Then, f is called a gradation preserving map (gp-map, in short) if for each $V \in I^Y$,

$$\delta(V) \leq \tau(f^{-1}(V)) \text{ and } \delta^*(V) \geq \tau^*(f^{-1}(V)).$$

Definition 2.7. (Abu Safia et al. [7]). Let X be a nonempty set and τ_1, τ_2 be two fuzzy topologies on X. Then (X, τ_1, τ_2) is called a bifuzzy topological space (BFTS, in short).

Definition 2.8. A fuzzy topological space (X, τ) is called

(a) T_0 if $\forall x, y \in X, x \neq y$, there exists $U \in \tau$ such that either $U(x) = 1, U(y) = 0$ or $U(y) = 1$, $U(x) = 0$.

(b) T_1 if $\forall x, y \in X$, $x \neq y$, there exist $U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$, $V(y) = 1$ and $V(x) = 1$.

(c) T_2 (Hausdorff) if $\forall$ pair of distinct fuzzy points x_r, y_s in X, there exist $U, V \in \tau$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = \underline{0}$.

Here, definitions (a), (b) and (c) are from [11], [12] and [10], respectively.

Definition 2.9. Let (X, τ_1, τ_2) be a BFTS. Then it is called

(a) T_0 if $\forall x, y \in X, x \neq y$, there exists $U \in \tau_1 \cup \tau_2$ such that $U(x) = 1, U(y) = 0$ or $U(x) = 0$, $U(y) = 1$.

(b) T_1 if $\forall x, y \in X, x \neq y$, there exist $U \in \tau_1$ and $V \in \tau_2$ such that $U(x) = 1$, $U(y) = 0$ and $V(x) = 0$, $V(y) = 1$.

(c) T_2 if $\forall$ pair of distinct fuzzy points x_r, y_s in X, there exist $U \in \tau_1$ and $V \in \tau_2$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = \underline{0}$.

Here definitions (a) and (b) are from [9] and (c) is from [13].

Definition 2.10. Let (X, τ, τ^*) be an IFTS and x_r be a fuzzy singleton in X. Fang [3] defined $Q_{x_r} : I^X \longrightarrow I$ as follows:

$$Q_{x_r}(U) = \begin{cases} \bigvee_{x_r q V \leq U} \tau(V), & if \quad x_r q U \\ 0 & if \quad x_r \neg q U \end{cases}$$

Here, $Q_{x_r}(U)$ is called the degree to which U is a Q-nbd of x_r.

We define $Q^*_{x_r} : I^X \longrightarrow I$ as follows:

$$Q^*_{x_r}(U) = \begin{cases} \bigwedge_{x_r q V \leq U} \tau^*(V), & if \quad x_r q U \\ 1 & if \quad x_r \neg q U \end{cases}.$$

$Q^*_{x_r}(U)$ will be called the degree to which U is a non Q-nbd of x_r.
We have,

$$Q_{x_r}(U) + Q^*_{x_r}(U) \leq 1, \forall U \in I^X.$$

3 α-T_0, α-T_1 and α-T_2 separation axioms in intuitionistic fuzzy topological spaces

Definition 3.1. Let (X, τ, τ^*) be an IFTS and x_r, y_s be two distinct fuzzy singletons in X. Then,

(a) The degree to which x_r, y_s are T_0 is defined as

$$T_0(x_r, y_s) = \Big(\bigvee_{y_s \neg q U} Q_{x_r}(U)\Big) \vee \Big(\bigvee_{x_r \neg q V} (1 - Q^*_{y_s}(V))\Big) \vee \Big(\bigvee_{x_r \neg q V} Q_{y_s}(V)\Big) \vee \Big(\bigvee_{y_s \neg q U} (1 - Q^*_{x_r}(U))\Big)$$

and the degree to which (X, τ, τ^*) is T_0, is defined as

$$T_0(X, \tau, \tau^*) = \bigwedge\{T_0(x_r, y_s) : x_r, y_s \text{ are distinct fuzzy singletons in X }\}.$$

(b) The degree to which x_r, y_s are T_1 is defined as

$$T_1(x_r, y_s) = \Big(\bigvee_{y_s \neg q U} Q_{x_r}(U)\Big) \wedge \Big(\bigvee_{x_r \neg q V} (1 - Q^*_{y_s}(V))\Big) \wedge \Big(\bigvee_{x_r \neg q V} Q_{y_s}(V)\Big) \wedge \Big(\bigvee_{y_s \neg q U} (1 - Q^*_{x_r}(U))\Big)$$

and the degree to which (X, τ, τ^*) is T_1 is defined as

$$T_1(X, \tau, \tau^*) = \bigwedge\{T_1(x_r, y_s) : x_r, y_s \text{ are distinct fuzzy singletons in X }\}.$$

(c) The degree to which x_r, y_s are T_2 is defined as

$$T_2(x_r, y_s) = \Big[\bigvee_{U \cap V = \underline{0}} \{Q_{x_r}(U) \wedge (1 - Q^*_{y_s}(V))\} \wedge \bigvee_{U \cap V = \underline{0}} \{Q_{y_s}(V) \wedge (1 - Q^*_{x_r}(U))\}\Big]$$

and the degree to which (X, τ, τ^*) is T_2 is defined as

$$T_2(X, \tau, \tau^*) = \bigwedge\{T_2(x_r, y_s) : x_r, y_s \text{ are distinct fuzzy singletons in X }\}.$$

It is easy to see that

(i) $T_2(X, \tau, \tau^*) \leq T_1(X, \tau, \tau^*) \leq T_0(X, \tau, \tau^*)$
but none of the implications are reversible.

(ii) If $\tau^*(U) = (1 - \tau(U))$, $\forall U \in I^X$ then the above definitions reduce to corresponding definitions in Yue and Fang [15].

Definition 3.2. An IFTS $(X, \tau, \tau*)$ is called

(a) α-T_0, $\alpha \in I_0$ (resp. α^*-T_0, $\alpha \in I_1$) if there exists $U \in I^X$ such that $\tau(U) \geq \alpha$, $\tau^*(U) \leq (1 - \alpha)$ (resp. $\tau(U) > \alpha$, $\tau^*(U) < (1 - \alpha)$) such that $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1, \forall\, x, y \in X, x \neq y$.

(b) α-T_1, $\alpha \in I_0$ (resp. α^*-T_1, $\alpha \in I_1$) if there exist $U, V \in I^X$ such that $\tau(U) \geq \alpha$, $\tau^*(V) \leq (1 - \alpha)$ (resp. $\tau(U) > \alpha$, $\tau^*(V) < (1 - \alpha)$) such that $U(x) = 1$, $U(y) = 0$ and $V(x) = 0$, $V(y) = 1, \forall\, x, y \in X, x \neq y$.

(c) α-T_2 (i.e. α-Hausdorff), $\alpha \in I_0$ (resp. α^*-T_2 i.e. α^*-Hausdorff, $\alpha \in I_1$) if $\forall$ distinct pair of fuzzy points x_r, y_s in X, there exist $U, V \in I^X$ such that $\tau(U) \geq \alpha$, $\tau^*(V) \leq (1-\alpha)$ (resp. $\tau(U) > \alpha$, $\tau^*(V) < (1-\alpha)$), $x_r \in U$, $y_s \in V$ and $U \cap V = \underline{0}$.

The following propositions can be easily verified.

Proposition 3.1. An IFTS (X, τ, τ^*) is α-T_i iff BFTS $(X, \tau_\alpha, \tau_\alpha^*)$ is T_i, $i = 0, 1, 2$.

Proposition 3.2. An IFTS (X, τ, τ^*) is α-Hausdorff, $\alpha \in I_0$ (resp. α^*-Hausdorff, $\alpha \in I_1$) iff $\forall$ distinct pair of fuzzy singletons x_r, y_s in X, there exist $U, V \in I^X$ such that $\tau(U) \geq \alpha$, $\tau^*(V) \leq (1-\alpha)$ (resp. $\tau(U) > \alpha$, $\tau^*(V) < (1-\alpha)$), $x_r qU$, $y_s qV$ and $U \cap V = \underline{0}$.

Proposition 3.3. If an IFTS (X, τ, τ^*) is α-T_i, $\alpha \in I_0$ (resp. α^*-T_i, $\alpha \in I_1$) then $T_i(X, \tau, \tau^*) \geq \alpha$, $i \in 0, 1, 2$.

Proof: Let us first suppose that (X, τ, τ^*) is α-T_0 then $(X, \tau_\alpha, \tau_\alpha^*)$ is T_0. Choose any two distinct fuzzy singletons x_r, y_s in X. Then $x \neq y$ and therefore there exists $U \in \tau_\alpha \cup \tau_\alpha^*$ such that $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1$. Let $U \in \tau_\alpha$ and be such that $U(x) = 1, U(y) = 0$. Then $\tau(U) \geq \alpha$, $x_r qU$, $y_s \neg qU \Rightarrow \bigvee_{y_s \neg qU} Q_{x_r}(U) \geq \alpha \Rightarrow T_0(x_r, y_s) \geq \alpha$ $\Rightarrow \bigwedge\{T_0(x_r, y_s) : x_r, y_s$ are distinct fuzzy singletons in $X\} \geq \alpha$ i.e. $T_0(X, \tau, \tau^*) \geq \alpha$. Now let $U \in \tau_\alpha$ be such that $U(x) = 0$, $U(y) = 1$. Then $\bigvee_{x_r \neg qU} Q_{y_s}(U) \geq \alpha \Rightarrow T_0(x_r, y_s) \geq \alpha$ $\Rightarrow T_0(X, \tau, \tau^*) \geq \alpha$. Further if $U \in \tau_\alpha^*$ and is such that $U(x) = 1, U(y) = 0 \Rightarrow Q_{x_r}^*(U) \leq (1-\alpha)$ $\Rightarrow (1 - Q_{x_r}^*(U)) \geq \alpha \Rightarrow \bigvee_{y_s \neg qU}(1 - Q_{x_r}^*)(U) \geq \alpha \Rightarrow T_0(x_r, y_s) \geq \alpha \Rightarrow T_0(X, \tau, \tau^*) \geq \alpha$ and if $U \in \tau_\alpha^*$ such that $U(x) = 0$, $U(y) = 1$ then $\bigvee_{x_r \neg qU}(1 - Q_{y_s}^*)(U) \geq \alpha \Rightarrow T_0(x_r, y_s) \geq \alpha$ $\Rightarrow T_0(X, \tau, \tau^*) \geq \alpha$.

Next, let (X, τ, τ^*) be α-T_1. Then, $(X, \tau_\alpha, \tau_\alpha^*)$ is T_1. Choose any pair of distinct fuzzy singletons x_r, y_s in X. Then $x \neq y$, hence there exist $U \in \tau_\alpha$, $V \in \tau_\alpha^*$ such that $U(x) = 1$, $U(y) = 0$, $V(x) = 0$, $V(y) = 1$. So we have $x_r qU$, $y_s \neg qU$, $x_r \neg qV$, $y_s qV$, $\tau(U) \geq \alpha$, $\tau^*(V) \leq (1-\alpha)$ $\Rightarrow \bigvee_{y_s \neg qU} Q_{x_r}(U) \geq \alpha$ and $Q_{y_s}^*(V) \leq (1-\alpha) \Rightarrow (1-Q_{y_s}^*(V)) \geq \alpha \Rightarrow \bigvee_{x_r \neg qV}(1-Q_{y_s}^*)(V) \geq \alpha$. Similarly, for the distinct pair of fuzzy singletons y_s and x_r in X, since $y \neq x$, there exist $U \in \tau_\alpha$, $V \in \tau_\alpha^*$ such that $U(x) = 0$, $U(y) = 1$, $V(x) = 1$, $V(y) = 0 \Rightarrow \bigvee_{x_r \neg qU} Q_{y_s}(U) \geq \alpha$ and $\bigvee_{y_s \neg qV}(1 - Q_{x_r}^*)(V) \geq \alpha$. Therefore, $T_1(x_r, y_s) \geq \alpha \Rightarrow \bigwedge\{T_1(x_r, y_s) : x_r, y_s$ are distinct fuzzy singletons in $X\} \geq \alpha$, i.e. $T_1(X, \tau, \tau^*) \geq \alpha$.

Finally, suppose that (X, τ, τ^*) is α-T_2. Then $(X, \tau_\alpha, \tau_\alpha^*)$ is T_2. Choose any pair of distinct fuzzy singletons x_r, y_s in X. Then there exist $U \in \tau_\alpha$, $V \in \tau_\alpha^*$ such that $x_r qU$, $y_s qV$ and $U \cap V = \underline{0}$. Hence

$$\bigvee_{U \cap V = \underline{0}} \{Q_{x_r}(U) \wedge (1 - Q_{y_s}^*(V))\} \geq \alpha.$$

Similarly considering the pair of fuzzy singletons y_s, x_r in X, there exist $U_1 \in \tau_\alpha$, $V_1 \in \tau_\alpha^*$ such that $y_s qU_1$, $x_r qV_1$ and $U_1 \cap V_1 = \underline{0}$. Therefore

$$\bigvee_{U_1 \cap V_1 = \underline{0}} \{Q_{y_s}(U_1) \wedge (1 - Q_{x_r}^*(V_1))\} \geq \alpha.$$

Thus $T_2(x_r, y_s) \geq \alpha \Rightarrow \bigwedge\{T_2(x_r, y_s) : x_r, y_s$ are distinct fuzzy singletons in $X\} \geq \alpha$ i.e. $T_1(X, \tau, \tau^*) \geq \alpha$.

On similar lines, it can be proved that if (X, τ, τ^*) is α^*-T_i then $T_i(X, \tau, \tau^*) \geq \alpha$, $i = 0, 1, 2$.

Proposition 3.4. The separation properties α-T_i (resp. α^*-T_i), $i = 0, 1, 2$ are hereditary.

The proof is easy and hence is omitted.

Theorem 3.1. Let $\{(X_i, \tau_i, \tau_i^*) : i \in J\}$ be a family of IFTSs. Then, their product IFTS (X, τ, τ^*) is α^*-T_1 iff (X_i, τ_i, τ_i^*) is α^*-T_1, $\forall i \in J$.

Proof: First let us suppose that each coordinate space (X_i, τ_i, τ_i^*) is α^*-T_1. To show that the product IFTS (X, τ, τ^*) is α^*-T_1, choose any two distinct points $x, y \in X$. Let $x = \Pi x_i$, $y = \Pi y_i$. Since $x \neq y$, there exist $j \in J$ such that $x_j \neq y_j$. Now since (X_j, τ_j, τ_j^*) is α^*-T_1, there exist $U_j, V_j \in I^{X_j}$ such that $\tau_j(U_j) > \alpha$, $\tau_j^*(V_j) < (1 - \alpha)$, $U_j(x_j) = 1$, $U_j(y_j) = 0$, $V_j(x_j) = 0$, $V_j(y_j) = 1$. Now consider $p_j^{-1}(U_j)$ and $p_j^{-1}(V_j)$. Since p_j is a gp-map, $\tau(p_j^{-1}(U_j)) > \alpha$, $\tau^*(p_j^{-1}(V_j)) < (1 - \alpha)$ and further, we have $p_j^{-1}(U_j)(x) = 1$, $p_j^{-1}(U_j)(y) = 0$, $p_j^{-1}(V_j)(y) = 1$, $p_j^{-1}(V_j)(x) = 0$. Hence (X, τ, τ^*) is α^*-T_1.
Conversely, let the product IFTS (X, τ, τ^*) be α^*-T_1. To show that (X_j, τ_j, τ_j^*) is α^*-T_1, choose any two distinct points x_j, y_j in X_j. Consider the distinct points $x = \Pi x_i$, $y = \Pi y_i$ in X where $x_i = y_i$ for $i \neq j$ and the j-th coordinate of x, y are x_j, y_j, respectively. Then there exist $U, V \in I^X$ such that $\tau(U) > \alpha$, $\tau^*(V) < (1 - \alpha)$, $U(x) = 1$, $U(y) = 0$, $V(x) = 0$, $V(y) = 1$. Now $\tau(U) = \wedge\{t : U \in T_t\} > \alpha$, $\tau^*(V) = \vee\{(1 - t) : V \in T_t^*\} \Rightarrow \exists\ t_1 > \alpha$ such that $U \in T_{t_1}$ and there exist $t_2 > \alpha$ such that $V \in T_{t_2}^*$. Now consider the distinct fuzzy points x_r and y_r. Then there exist basic fuzzy open sets ΠU_i^r and ΠV_i^r in T_{t_1} and T_{t_2} respectively such that $x_r \in \Pi U_i^r \subseteq U$ and $y_r \in \Pi V_i^r \subseteq V$. Hence, $r < \Pi U_i^r(x) \leq U(x)$, $r < \Pi V_i^r(y) \leq V(y)$. Therefore,

$$r < inf\{U_{k_1}^r(x_{k_1}), U_{k_2}^r(x_{k_2}), ..., U_{k_m}^r(x_{k_m})\} \tag{1}$$

and

$$r < inf\{V_{l_1}^r(y_{l_1}), V_{l_2}^r(y_{l_2}), ..., V_{l_n}^r(y_{l_n})\}. \tag{2}$$

Now we claim that

$$j \in \{k_1, k_2, ..., k_m\} \wedge \{l_1, l_2, ..., l_n\}. \tag{3}$$

Since if it is not so, then $x_{l_1} = y_{l_1}, x_{l_2} = y_{l_2}, ..., x_{l_n} = y_{l_n}$ and hence in view of (2),

$$r < inf\{V_{l_1}^r(x_{l_1}), V_{l_2}^r(x_{l_2}), ..., V_{l_n}^r(x_{l_n})\}$$

Therefore, $\Pi V_i^r(x) > 0$. Hence, $V(x) > 0$, which is a contradiction. Similarly, it can be shown that $U(y) > 0$, a contradiction. Thus, $U_j^r(x_j) > r$ and $V_j^r(y_j) > r$ implying that $(x_j)_r \in U_j^r$, $(y_j)_r \in V_j^r$. Now, consider $U_j = \cup_{r \in I_0} U_j^r$, $V_j = \cup_{r \in I_0} V_j^r$.Then, $U_j(x_j) = 1$, $V_j(y_j) = 1$. Now, it remains to show that $U_j(y_j) = 0$, $V_j(x_j) = 0$. Since $U(y) = 0$, $\Pi U_i^r(y) = 0 \Rightarrow$ inf $\{U_{k_1}^r(y_{k_1}), U_{k_2}^r(y_{k_2}), ..., U_{k_m}^r(y_{k_m})\} = 0 \Rightarrow U_j^r(y_j) = 0$ in view of (1), (3) and the fact that $x_i = y_i$ for $i \neq j$, $\forall r \in I_0$. Hence, $U_j(y_j) = \sup U_j^r(y_j) = 0$. Similarly, it can be shown that $V_j(x_j) = 0$. Further $\forall r \in I_0$, $\tau_j(U_j^r) \geq t_1 > \alpha$ and $\tau_j^*(V_j^r) \leq (1 - t_2) < (1 - \alpha)$. Therefore, $\tau_j(\cup_r U_j^r) \geq \wedge_r \tau_j(U_j^r) \geq t_1 > \alpha$ and $\tau_j^*(\cup_r V_j^r) \leq \vee_r \tau_j^*(V_j^r) \leq (1 - t_2) < (1 - \alpha)$.
Hence, $\tau_j(U_j) > \alpha$ and $\tau_j^*(V_j) < (1 - \alpha)$. Thus, (X, τ_j, τ_j^*) is α^*-T_1.

The following theorem can be proved on similar lines.

Theorem 3.2. Let $\{(X_i, \tau_i, \tau_i^*) : i \in J\}$ be a family of IFTSs. Then, their product IFTS (X, τ, τ^*) is α^*-T_0 iff each coordinate space is α^*-T_0.

Theorem 3.3. Let $\{(X_i, \tau_i, \tau_i^*) : i \in J\}$ be a family of IFTSs. Then their product IFTS (X, τ, τ) is α^*-Hausdorff iff each coordinate space (X_i, τ_i, τ_i^*) is α^*-Hausdorff.

Proof: Let each coordinate space (X_i, τ_i, τ_i^*) be α^*-Hausdorff. Then to show that the product IFTS (X, τ, τ^*) is α^*-Hausdorff, consider any two distinct fuzzy points x_r, y_s in X. Then $x \neq y$. Let $x = \Pi x_i$ and $y = \Pi y_i$ then there exists $j \in J$ such that $x_j \neq y_j$. Now, consider the distinct fuzzy points $(x_j)_r$ and $(y_j)_s$ in X_j. Since (X_j, τ_j, τ_j^*) is α^*-Hausdorff, there exist $U_j, V_j \in I^{X_j}$ such that $\tau_j(U_J) > \alpha$, $\tau_j^*(V_j) < (1-\alpha)$ and $(x_j)_r \in U_j$, $(y_j)_s \in V_j$ and $U_j \cap V_j = \underline{0}$.

Let $U = p_j^{-1}(U_j)$ and $V = p_j^{-1}(V_j)$. Then since p_j is a gp-map, $\tau(U) \geq \tau_j(U_j) > \alpha$ and $\tau^*(V) \leq \tau_j^*(V_j) < (1-\alpha)$. Further, $x_r \in p_j^{-1}(U_j)$, $y_s \in p_j^{-1}(V_j)$, $p_j^{-1}(U_j) \cap p_j^{-1}(V_j) = \underline{0}$. Hence, (X, τ, τ^*) is α^*-Hausdorff.
Conversely, let (X, τ, τ^*) be α^*-Hausdorff. To show that (X_j, τ_j, τ_j^*) is α^*-Hausdorff, choose any two distinct fuzzy points $(x_j)_r$, $(y_j)_s$ in X_j.Then, $x_j \neq y_j$. Consider $x = \Pi x_i$, $y = \Pi y_i$ where $x_i = y_i$ for $i \neq j$ and the j^{th} coordinate of x, y are x_j and y_j respectively. Consider the distinct fuzzy points x_r and y_s in X. Since (X, τ, τ^*) is α^*-Hausdorff, there exist $U, V \in I^X$ such that $\tau(U) > \alpha$, $\tau^*(V) < (1-\alpha)$, $x_r \in U$, $y_s \in V$ and $U \cap V = \underline{0}$.
Now $\tau(U) = \vee\{t : U \in T_t\} > \alpha$ and $\tau^*(V) = \wedge\{(1-t) : V \in T_t^*\} < (1-\alpha)$ which implies that there exists $t_1 > \alpha$ such that $U \in T_{t_1}$ and there exists $t_2 > \alpha$ such that $V \in T_{t_2}^*$. Since $U \in T_{t_1}$ and $x_r \in U$, there exists a basic fuzzy open set

$$W_1 = p_{k_1}^{-1}(U_{k_1}) \cap p_{k_2}^{-1}(U_{k_2}), ..., \cap p_{k_m}^{-1}(U_{k_m})$$

in T_{t_1} such that $x_r \in W_1 \subseteq U$ which implies that

$$r < inf\{p_{k_1}^{-1}(U_{k_1})(x), p_{k_2}^{-1}(U_{k_2})(x), ..., p_{k_m}^{-1}(U_{k_m})(x)\}$$

i.e.

$$r < inf\{U_{k_1}(x_{k_1}), U_{k_2}(x_{k_2}), ..., U_{k_m}(x_{k_m})\} \tag{4}$$

Similarly since $y_s \in V$ and $V \in T_{t_2}^*$, there exists a basic fuzzy open set

$$W_2 = p_{l_1}^{-1}(V_{l_1}) \cap p_{l_2}^{-1}(V_{l_2}), ..., \cap p_{l_n}^{-1}(V_{l_n})$$

in $T_{t_2}^*$ such that $y_s \in W_2 \subseteq V$ which implies that

$$s < inf\{p_{l_1}^{-1}(V_{l_1})(y), p_{l_2}^{-1}(V_{l_2})(y), ..., p_{l_n}^{-1}(V_{l_n})(y)\}$$

i.e.

$$s < inf\{V_{l_1}(y_{l_1}), V_{l_2}(y_{l_2}), ..., V_{l_n}(y_{l_n})\} \tag{5}$$

Now we claim that $j \in \{k_1, k_2, ..., k_m\} \cap \{l_1, l_2, ..., l_n\}$. Since if it not so, then $x_{l_1} = y_{l_1}, x_{l_2} = y_{l_2}, ..., x_{l_m} = y_{l_m}$. Hence, in view of (5), we have $s < \{V_{l_1}(x_{l_1}), V_{l_2}(x_{l_2}), ..., V_{l_n}(x_{l_n})\} \Rightarrow W_2(x) > 0 \Rightarrow V(x) > 0 \Rightarrow U \cap V(x) > 0$. which is a contradiction to the fact that $U \cap V = \underline{0}$. Hence, $U_j(x_j) > r$, $V_j(y_j) > s \Rightarrow (x_j)_r \in U_j, (y_j)_s \in V_j$. Now we show that $U_j \cap V_j = \underline{0}$. If $U_j \cap V_j \neq \underline{0}$, there exists $z_j \in X_j$ such that

$$U_j(z_j) > 0, V_j(z_j) > 0. \tag{6}$$

Now, consider $z = \Pi z_i$ where $z_i = x_i = y_i$ for $i \neq j$ and the j-th coordinate is z_j. Then in view of (4), (5) and (6) we get $W_1(z) > 0$, $W_2(z) > 0$ which implies that $W_1 \cap W_2 \neq \underline{0}$. Therefore, $U \cap V \neq \underline{0}$, again a contradiction. Hence, $U_j \cap V_j = \underline{0}$. Further, $\tau_j(U_j) \geq t_1 > \alpha$ and $\tau_j^*(V_j) \leq (1-t_2) < (1-\alpha)$ showing that (X_j, τ_j, τ_j^*) is α^*-Hausdorff.

Proposition 3.5. Let $\{(X_j, \tau_j, \tau_j^*) : j \in J\}$ be a family of IFTSs, (X, τ, τ^*) be their product IFTS. Let T_t denote the product fuzzy topology $\Pi(T_j)_t$ and let T_t^* denote the product fuzzy topology $\Pi(T_j)_t^*$ on X. Then,

$$(i)\ \bigcap_{s<r} T_s = T_r.$$

$$(ii)\ \bigcap_{s<r} T^*_s = T^*_r.$$

Proof:

(i) Since $T_r \subset T_s$, for all $s < r$, we have $T_r \subseteq \bigcap_{s<r} T_s$.
Conversely, $U \in \bigcap_{s<r} T_s \Rightarrow U \in T_s, \forall\ s < r$. Hence,
$\tau(U) = \vee\{t : U \in T_t\} \geq s$, for all $s < r$ i.e., $\tau(U) \geq r \Rightarrow U \in T_r$
Therefore, $\bigcap_{s<r} T_s \subseteq T_r$. Thus, $\bigcap_{s<r} T_s = T_r$.

(ii) $T^*_r \subseteq T^*_s$, for all $s < r$, $\Rightarrow T^*_r \subseteq \bigcap_{s<r} T^*_s$
Conversely, let $V \in \bigcap_{s<r} T^*_s \Rightarrow V \in T^*_s$, for all $s < r$. Hence,
$\tau^*(V) = \bigvee\{(1-t) : V \in T^*_t\} \leq (1-s)$, for all $s < r$ i.e. $\tau^*(V) \leq (1-r) \Rightarrow V \in T^*_r$.
Therefore, $\bigcap_{s<r} T^*_s \subseteq T^*_r$. Thus $\bigcap_{s<r} T^*_s = T^*_r$.

Theorem 3.4. If $\{(X_j, \tau_j, \tau^*_j) : j \in J\}$ is a family of IFTSs and (X, τ, τ^*) is their product IFTS. Then $\tau_r = \Pi(\tau_j)_r$, $\tau^*_r = \Pi(\tau^*_j)_r$.

The proof follows from Theorem 2.15, Definition 5.5 of (Mondal and Samanta [6]) and the previous proposition.

Theorem 3.5. Let $\{(X_j, \tau_j, \tau^*_j) : j \in J\}$ be a family of IFTSs and (X, τ, τ^*) be their product IFTS. Then, (X, τ, τ^*) is α-T_i iff each coordinate space (X_j, τ_j, τ^*_j) is α-T_i, $i = 0, 1, 2$.

Proof: (X, τ, τ^*) is α-$T_i \Leftrightarrow (X, \tau_\alpha, \tau^*_\alpha)$ is T_i
$\Leftrightarrow (X, \Pi(\tau_j)_\alpha, \Pi(\tau_j)^*_\alpha)$ is T_i
$\Leftrightarrow (X_j, (\tau_j)_\alpha, (\tau_j)^*_\alpha)$ is T_i, $\forall\ j \in J$
$\Leftrightarrow (X_j, \tau_j, \tau^*_j)$ is α-T_i, $\forall\ j \in J$, $i = 0, 1, 2$.

References

[1] Atanassov, K.T. Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] Chang, C.L. Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[3] Fang, J.I-FTOP is isomorphic to I-FQN and I-AITOP, Fuzzy Sets and Systems 147 (2004) 317–325.

[4] Kubiak, T. On fuzzy topologies, Ph.D Thesis, A. Mickiewicz, Poznan, Poland, 1985.

[5] Pu, P.-M. Y.-M. Liu. Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571-599.

[6] Mondal, T.K. , S.K. Samanta. On intuitionistic gradation of openness, Fuzzy Sets and Systems 131 (2002) 323–336.

[7] Abu Safiya, A.S. , A.A. Fora, M.W. Warner. Fuzzy separation axioms and fuzzy continuity in fuzzy bitopological spaces, Fuzzy Sets and Systems 62 (1994) 367–373.

[8] Šostak, A.P. On fuzzy topological structure, Rend. Circ. Mat. Palermo (Suppl. Ser. II) 11 (1985) 89–103.

[9] Srivastava, M., R. Srivastava. On fuzzy pairwise - T_0 and fuzzy pairwise - T_1 bitopological spaces, Indian J. Pure Appl. Math. 32 (2001) 387–396.

[10] Srivastava, R., S.N. Lal, and A.K. Srivastava. Fuzzy Hausdorff topological spaces, J. Math. Anal. Appl. 81 (1981) 497-506.

[11] Srivastava, R., S.N. Lal, A.K. Srivastava. On fuzzy T_0 and R_0 topological spaces, J. Math. Anal. Appl. 136 (1988) 66–73.

[12] Srivastava, R., S.N. Lal, A.K. Srivastava.On fuzzy T_1- topological spaces, J. Math. Anal. Appl. 136 (1988) 124–130.

[13] Srivastava, R., M. Srivastava. On pairwise Hausdorff fuzzy bitopological spaces, J. Fuzzy Math. 5 (1997) 553–564.

[14] Wong, C.K. Fuzzy points and local properties of fuzzy topology, J. Math. Anal. Appl. 46 (1974) 316–328.

[15] Yue,Y., J. Fang. On separation axioms in I-fuzzy topological spaces, Fuzzy Sets and Systems 157 (2006) 780–793.

[16] Zadeh, L.A. Fuzzy sets, Inform. and Control 8 (1965) 338–353.

[17] Zadeh, L.A. A fuzzy set theoretic interpretation of linguistic hedges, Memorandum No. ERLM335 University of California, Berkeley (1972).