Taylor's theorem for functions defined on Atanassov *IF*-sets

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IF sets

$A = (\mu_A, \nu_A)$ - *IF* set

$$\mu_A(x), \nu_A(x) : \Omega \to [0, 1]$$

$$\mu_A(x) + \nu_A(x) \leq 1, \forall x \in A.$$

- $\mu_A(x)$ - membership function,
- $\nu_A(x)$ - nonmembership function

IF events

$A = (\mu_A, \nu_A)$ - *IF* event

- $(\Omega, \mathcal{S})$ - measurable space,
- μ_A, ν_A - $\mathcal{S}$-measurable.

$\mathcal{F}$ - the family of all *IF* events.

IF sets - some operations and relations

Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, then

$$\begin{aligned}
A + B &= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0) \\
A - B &= ((\mu_A - \mu_B) \vee 0, (\nu_A - \nu_B + 1) \wedge 1) \\
A.B &= (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B) \\
A \vee B &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B) \\
A \wedge B &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \\
A \leq B &\Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.
\end{aligned}$$

Theorem

We could construct such ℓ-group G that $\mathcal{F}$ can be embedded into G.

- Hollá I., Riečan B: Elementary function on *IF* sets. *Issues in Soft Computing.* Warszawa, 2008, Akademicka Oficyna Wydawnicza EXIT.

ℓ-group G

$G = (G, +, \leq)$ - ℓ-group

- $(G, +)$ - Abelian group
- $(G, \leq)$ - partially ordered set
- $a \leq b \quad \Rightarrow \quad a + c \leq b + c$

ℓ-group G

- $A = (\mu_A, \nu_A), \ \ \mu_A, \nu_A : \Omega \to R$,
- G - the set of all pairs $A = (\mu_A, \nu_A)$,
- for $A, B \in G$ define the operation and relation

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B - 1)$$

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Then the triple $G = (G, +, \leq)$ is the mentioned ℓ-group.

Definition of the function

Let f be a real function. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $A \leq B$ and $[\mu_A, \mu_B] \cup [\nu_B, \nu_A] \subset Dom\, f$.

Then the function $\bar{f} : [A, B] \to R^2$ is defined by

$$\bar{f}(X) = (f(\mu_X), 1 - f(1 - \nu_X))$$

where $X = (\mu_X, \nu_X)$ is called a variable.

- A. Michalíková. Some notes about boundaries on IF sets. *New Trends in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized nets and Related Topics.* Poland, 2013.

Example

Since

$$\bar{f}(X) = (f(\mu_X), 1 - f(1 - \nu_X))$$

then for example

$$X^n = ((\mu_X)^n, 1 - (1 - \nu_X)^n), n \in N.$$

Definition of the derivative of the function

Let f be a real function. Let $X = (\mu_X, \nu_X)$ be a variable.

Then

$$\bar{f}'(X) = \big(f'(\mu_X), 1 - f'(1 - \nu_X)\big)$$

is the derivative of the function on the family of all *IF*-sets.

- A. Michalíková. The differential calculus on IF sets. *FUZZ-IEEE 2009.* [CD-ROM]. Jeju Island, Korea, 2009, p. 1393-1395, ISBN: 978-1-4244-3597-5, ISSN 1098-7584.

Definition of the polynomial function

Let $n \in N$, $X = (\mu_X, \nu_X)$ be the variable and $A_i \in R^2$, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 0, 1, 2, \ldots, n$ be the constants, $A_n \neq (0, 1)$.

Then

$$\bar{p}_n(X) = A_0 + A_1 X + A_2 X^2 + \ldots + A_n X^n$$

is called the polynomial function.

Theorem

Let $X = (\mu_X, \nu_X)$ be the variable and $\bar{p}_n$ be a polynomial function.

Then it holds

$$\bar{p}_n((\mu_X, \nu_X)) = (\mu_{A_0} + \mu_{A_1}\mu_X + \mu_{A_2}\mu_X^2 + \ldots + \mu_{A_n}\mu_X^n,$$

$$\nu_{A_0} + (\nu_{A_1} - 1)(1 - \nu_X) + (\nu_{A_2} - 1)(1 - \nu_X)^2 + \ldots + (\nu_{A_n} - 1)(1 - \nu_X)^n).$$

Theorem

Let $\overline{f}$ be continuous on $[A, B]$ and differentiable on (A, B) then there exists $C \in (A, B)$ such that

$$\overline{f}(B) - \overline{f}(A) = (f(\mu_B - \mu_A), f(1 - \nu_A) - f(1 - \nu_B) + 1) =$$

$$= (f'(\mu_C)(\mu_B - \mu_A), f'(1 - \nu_C)(\nu_B - \nu_A) + 1) = \overline{f}'(C)(B - A).$$

- B. Riečan. On Largange mean value theorem for functions on Atanassov *IF* sets. *Proceedings of the Eighth International Workshop on Intuitionistic Fuzzy Sets.* Slovakia. Vol. 18, 2012, p. 8-11, ISSN-1310-4926.

Definition

Let $X = (\mu_X, \nu_X)$ be the variable, let $X_0 = (\mu_{X_0}, \nu_{X_0})$ be the fixed point. Let $\bar{f}(X) = (f(\mu_X), 1 - f(1 - \nu_X))$ be the function. Let the derivations $\bar{f}^{(i)}(X) = (f^{(i)}(\mu_X), 1 - f^{(i)}(1 - \nu_X))$ exist for $i = 1, 2, \ldots, n$.

Then the Taylor's formula at the point X_0 has the following form

$$\bar{T}_n(X) = \bar{f}(X_0) + \frac{\bar{f}^{(1)}(X_0)}{1!}(X - X_0) + \ldots + \frac{\bar{f}^{(n)}(X_0)}{n!}(X - X_0)^n.$$

Theorem

Function $\bar{T}_n$ is the Taylor's formula at the point $X_0 = (\mu_{X_0}, \nu_{X_0})$ if and only if for any $X = (\mu_X, \nu_X)$ it holds

$$\bar{T}_n((\mu_X, \nu_X)) =$$

$$= \Big(f(\mu_{X_0}) + \frac{f^{(1)}(\mu_{X_0})}{1!}(\mu_X - \mu_{X_0}) + \ldots + \frac{f^{(n)}(\mu_{X_0})}{n!}(\mu_X - \mu_{X_0})^n,$$

$$1 - (f(1 - \nu_{X_0}) + \frac{f^{(1)}(1 - \nu_{X_0})}{1!}(\nu_{X_0} - \nu_X) + \ldots + \frac{f^{(n)}(1 - \nu_{X_0})}{n!}(\nu_{X_0} - \nu_X)^n) \Big).$$

Example

Let $\bar{f}(X) = \overline{\sin}(X) = (\sin(\mu_X), 1 - \sin(1 - \nu_X))$ and $X_0 = (0, 1)$.

Then

$$\bar{T}_n((\mu_X, \nu_X)) =$$

$$= \left(\frac{1}{1!}\mu_X - \frac{1}{3!}\mu_X^3 + \ldots + \frac{(-1)^n}{(2n+1)!}\mu_X^{2n+1}, \right.$$

$$\left. 1 - \left(\frac{1}{1!}(1 - \nu_X) - \frac{1}{3!}(1 - \nu_X)^3 + \ldots + \frac{(-1)^n}{(2n+1)!}(1 - \nu_X)^{2n+1} \right) \right).$$

Taylor's theorem

Let $\bar{f}$ be a function that has continuous derivations $\bar{f}^{(i)}$ defined on interval $[X_0, X]$, let there exists derivation $\bar{f}^{(n+1)}$ on interval (X_0, X) and $\bar{T}_n$ be the Taylor's formula appertaining to $\bar{f}$ in the point X_0.

Then there exist such function $\bar{R}_n$ and such $C = (\mu_C, \nu_C)$, $C \in (X_0, X)$ that it holds

$$\bar{f}(X) = \bar{T}_n(X) + \bar{R}_n(X).$$

Remainder

In the formula

$$\bar{f}(X) = \bar{T}_n(X) + \bar{R}_n(X)$$

the function $\bar{R}_n$ is usually called remainder and it could have following form

$$\bar{R}_n(X) = \frac{\bar{f}^{(n+1)}(C)}{(n+1)!}(X - X_0)^{n+1}$$

(Lagrange's form).

Thank you
for your attention!