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$(\in, \in \vee q)$-Intuitionistic fuzzy prime ideals of BCK–algebras

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Abstract: In this paper, we introduced the concept of $(\in, \in \vee q)$-intuitionistic fuzzification of prime ideals in commutative BCK-algebras. We state and proved some theorems in $(\in, \in \vee q)$-intuitionistic fuzzy prime ideals in commutative BCK-algebras. Characterization of $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal and conditions for $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal to be an $(\in, \in)$-intuitionistic fuzzy prime ideal are provided. Relation between $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal and $(\in \vee q)$-intuitionistic fuzzy level prime ideal were discussed.
Keywords: BCK-algebra, Prime ideal, Fuzzy prime ideal,$(\in, \in \vee q)$-Intuitionistic fuzzy prime ideal.
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1 Introduction

The concept of fuzzy sets was first initiated by Zadeh [20] in 1965. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, vector spaces, etc. Imai and Iseki [10] introduced BCK-algebras as a generalization of notion of the concept of set theoretic difference and propositional calculus and in the same year Iseki [11] introduced the notion of BCI-algebra.

It is known that the class of BCK-algebra is a proper sub class of the class of BCI-algebras. Iseki [12] introduced the concept of prime ideal in commutative BCK-algebras. In [2], Ahsan et al. studied the theory of ideals in particular, prime ideals of a commutative BCK-algebras. Completely fuzzy prime ideals of distributed lattices was studied by Attallah [4] and Koguep et al. [13] discussed fuzzy ideals in hyperlattices. The concept of prime fuzzy ideal was first introduced by Swamy and Raju [16]. In [17, 18], Jun and Xin have studied fuzzy prime ideals and invertible fuzzy ideals in BCK-algebras. Abdullah [1] introduced the notion of intuitionistic fuzzy prime ideals of commutative BCK-algebras. The concept of fuzzy point introduced by Ming and Ming in [14] and also they introduced the idea of relation "belongs to" and "quasi coincident with" between fuzzy point and fuzzy set. Murali [15] proposed a definition of a fuzzy point belonging to fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das [8, 9] used the relation of "belongs to" and "quasi-coincident" between fuzzy point and fuzzy set to introduced the concept of $(\in, \in \vee q)$-fuzzy subgroup, $(\in, \in \vee q)$-fuzzy subring and $(\in \vee q)$-level subset. Basnet and Singh [5] introduced $(\in, \in \vee q)$-fuzzy ideals of BG-algebra in 2011 and some properties of $(\in, \in \vee q)$-fuzzy ideals of d-algebra was discussed by Barbhuiya and Choudhury [6]. In [7], Barbhuiya introduced $(\in, \in \vee q)$-intuitionistic fuzzy ideals of BCK/BCI-algebras. It is now natural to investigate similar type of generalisation of the existing fuzzy subsystem with other algebraic structure. In this paper, we introduced the notion of $(\in, \in \vee q)$-intuitionistic fuzzy prime ideals of commutative BCK-algebras and got some interesting results.

2 Preliminaries

Definition 2.1. [1, 17, 18] An algebra $(X, *, 0)$ of type (2, 0) is called a BCK-algebra if it satisfies the following axioms:
(i) $((x * y) * (x * z)) * (z * y) = 0$;
(ii) $(x * (x * y)) * y = 0$;
(iii) $x * x = 0$;
(iv) $0 * x = 0$;
(v) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y, z \in X$.
We can define a partial ordering " $\leq$ " on X by $x \leq y$ iff $x * y = 0$

Definition 2.2. [1, 17, 18] A BCK-algebra X is said to be commutative if it satisfies the identity $x \wedge y = y \wedge x$ where $x \wedge y = y * (y * x)$ $\forall x, y \in X$. In a commutative BCK-algebra, it is known that $x \wedge y$ is the greatest lower bound of x and y.

In a BCK-algebra X, the following hold:
(i) $x * 0 = x$;
(ii) $(x * y) * z = (x * z) * y$;
(iii) $x * y \leq x$;
(iv) $(x * y) * z \leq (x * z) * (y * z)$;
(v) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Definition 2.3. [19] A nonempty subset I of a BCK-algebra X is called an ideal of X if

(i) $0 \in I$;
(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$ for all $x, y \in X$.

Definition 2.4. [19] A fuzzy set μ in BCK-algebra X is called a fuzzy ideal of X if it satisfies the following axioms:
(i) $\mu(0) \geq \mu(x)$;
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Definition 2.5. [1] An ideal I of a commutative BCK-algebra X is said to be prime if $x \wedge y \in I \Rightarrow x \in I$ or $x \in I$.

Definition 2.6. [18] A non-constant fuzzy ideal μ of a commutative BCK-algebra X is said to be fuzzy prime if $\mu(x \wedge y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Since $x \wedge y \leq x, y$ and μ is order reversing, it follows that $\mu(x) \leq \mu(x \wedge y)$ and $\mu(y) \leq \mu(x \wedge y)$ Therefore, a non-constant fuzzy ideal μ of a commutative BCK-algebra X is fuzzy prime iff $\mu(x \wedge y) = \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$ or equivalently $\mu(x \wedge y) = \mu(x)$ or $\mu(y)$ for all $x, y \in X$.

Definition 2.7. [3] An intuitionistic fuzzy set (IFS) A in a non-empty set X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non membership of the element x in the set A. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$.

Definition 2.8. [3,7] If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$ be any two IFS of a set X then: $A \subseteq B$ iff for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$; $A = B$ iff for all $x \in X, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$; $A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x)\rangle | x \in X\}$, where $(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$; $A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle | x \in X\}$, where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

Definition 2.9. [7] An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra X is said to be an intuitionistic fuzzy ideal of X if
(i) $\mu_A(0) \geq \mu_A(x)$
(ii) $\nu_A(0) \leq \nu_A(x)$
(iii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$
(iv) $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\} \quad \forall x, y \in X$.

Definition 2.10. [1] An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra X is called an intuitionistic fuzzy prime ideal of X if
(i) $\mu_A(x \wedge y) \leq \max\{\mu_A(x), \mu_A(y)\}$
(ii) $\nu_A(x \wedge y) \geq \min\{\nu_A(x), \nu_A(y)\} \quad \forall\, x, y \in X$.

Definition 2.11. [8] A fuzzy set μ of the form

$$\mu(y) = \begin{cases} t & \text{if} \quad y = x, \quad t \in (0, 1] \\ 0 & \text{if} \quad y \neq x \end{cases}$$

is called a fuzzy point with support x and value t and it is denoted by x_t.

Definition 2.12. [8] Let μ be a fuzzy set in X and x_t be a fuzzy point then
(i) If $\mu(x) \geq t$ then we say x_t belongs to μ and write $x_t \in \mu$.
(ii) If $\mu(x) + t > 1$ then we say x_t quasi coincidence with μ and write $x_t q\mu$.
(iii) If $x_t \in \vee q\mu \Leftrightarrow x_t \in \mu$ or $x_t q\mu$.
(iv) If $x_t \in \wedge q\mu \Leftrightarrow x_t \in \mu$ and $x_t q\mu$.

The symbol $x_t\overline{\alpha}\mu$ means $x_t\alpha\mu$ does not hold and $\overline{\in \wedge q}$ means $\overline{\in} \vee \overline{q}$.

For a fuzzy point x_t and a fuzzy set μ in set X, Pu and Liu [14] gave meaning to the symbol $x_t\alpha\mu$ where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 2.13. [19] A fuzzy set μ of a BCK-algebra X is said to be (α, β)-fuzzy ideal of X, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$ if
(i) $x_t\alpha\mu \Rightarrow 0_t\beta\mu$;
(ii) $(x * y)_t, y_s\alpha\mu \Rightarrow x_{m(t,s)}\beta\mu$ for all $x, y \in X$, where $t, s \in (0, 1]$.

Example 2.14. Consider BCK-algebra $X = \{0, x, y, z\}$ with the following Cayley table.

$*$	0	x	y	z	w
0	0	0	0	0	0
x	x	0	x	0	x
y	y	y	0	y	0
z	z	x	z	0	z
w	w	w	y	w	0

Define a map $\mu : X \to [0, 1]$ by $\mu(0) = 0.7, \mu(x) = \mu(z) = 0.3, \mu(y) = \mu(w) = 0.2$, then μ is an $(\in, \in \vee q)$-fuzzy ideal X.

Definition 2.15. [7] A fuzzy point x_t is said to belong to (respectively, be quasi coincident with) an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ written as $x_t \in A$ (respectively, $x_t qA$) if $\mu_A(x) \geq t$ (respectively, $\mu_A(x) + t > 1$ and $\nu_A(x) < t$ (respectively, $\nu_A(x) + t \leq 1$). If $x_t \in A$ or $x_t qA$, then we write $x_t \in \vee qA$.

Note: $x_t \in A \Rightarrow x_t \in \mu_A$ and $x_t\overline{\in}\nu_A$ and $x_t qA \Rightarrow x_t q\mu_A$ and $x_t\overline{q}\nu_A$.

Definition 2.16. [7] An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BCK-algebra X is said to be an (α, β)-intuitionistic fuzzy ideal of X, Where $\alpha \neq \in \wedge q$ if
(i) $x_t\alpha\mu_A \Rightarrow 0_t\beta\mu_A$;
(ii) $x_t\overline{\alpha}\nu_A \Rightarrow 0_t\overline{\beta}\nu_A$;
(iii) $(x * y)_t, y_s\alpha\mu_A \Rightarrow x_{m(t,s)}\beta\mu_A$ where $m(t, s) = \min(t, s)$ and $t, s \in (0, 1]$;
(iv) $(x * y)_t, y_s\overline{\alpha}\nu_A \Rightarrow x_{M(t,s)}\overline{\beta}\nu_A$ where $M(t, s) = \max(t, s)$ and $t, s \in (0, 1]$ for all $x, y \in X$.

Definition 2.17. An (α, β)-fuzzy ideal μ of a BCK-algebra X is said to be (α, β)-fuzzy prime ideal of X, Where $\alpha \neq \in \wedge q$ if $(x \wedge y)_t\alpha\mu \Rightarrow x_t\beta\mu$ or $y_t\beta\mu$ for all $x, y \in X$ where $t, s \in (0, 1]$.

Definition 2.18. An $(\in, \in \vee q)$-fuzzy ideal μ of a BCK-algebra X is said to be $(\in, \in \vee q)$-fuzzy prime ideal of X if $(x \wedge y)_t \in \mu \Rightarrow x_t \in \vee q\mu$ or $y_t \in \vee q\mu$ for all $x, y \in X$, where $t \in (0, 1]$.

3 $(\in, \in \vee q)$-Intuitionistic fuzzy prime ideals of BCK-algebra

Now onwards, let X denote a commutative BCK-algebra, unless otherwise stated.

Definition 3.1. An $(\in, \in \vee q)$-intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of a BCK-algebra X is said to be an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X if
(i) $(x \wedge y)_t \in \mu_A \Rightarrow x_t \in \vee q \mu_A$ or $y_t \in \vee q \mu_A$;
(ii) $(x \wedge y)_s \overline{\in} \nu_A \Rightarrow x_s \overline{\in \vee q} \nu_A$ or $y_s \overline{\in \vee q} \nu_A$ for all $x, y \in X$, where $t, s \in (0, 1]$.

Theorem 3.2. *An intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of a BCK-algebra X is an intuitionistic fuzzy prime ideal if and only if A is an $(\in, \in)$-intuitionistic fuzzy prime ideal.*

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy prime ideal. Therefore,

$\mu_A(x \wedge y) \leq \max\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \wedge y) \geq \min\{\nu_A(x), \nu_A(y)\}$. Let
$(x \wedge y)_t \in \mu_A \Rightarrow \mu_A(x \wedge y) \geq t$
$\Rightarrow \max\{\mu_A(x), \mu_A(y)\} \geq \mu_A(x \wedge y) \geq t$
$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t$
$\Rightarrow x_t \in \mu_A$ or $y_t \in \mu_A$

$$(x \wedge y)_t \in \mu_A \Rightarrow x_t \in \mu \text{ or } y_t \in \mu. \tag{3.1}$$

Again, let
$(x \wedge y)_t \overline{\in} \nu_A \Rightarrow \nu_A(x \wedge y) < t$
$\Rightarrow \min\{\nu_A(x), \nu_A(y)\} \leq \nu_A(x \wedge y) < t$
$\Rightarrow \nu_A(x) < t$ or $\nu_A(y) < t$
$\Rightarrow x_t \overline{\in} \nu_A$ or $y_t \overline{\in} \nu_A$

$$(x \wedge y)_t \overline{\in} \nu_A \Rightarrow x_t \overline{\in} \nu \text{ or } y_t \overline{\in} \nu. \tag{3.2}$$

Therefore, from equations (3.1) and (3.2), $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X.

Conversely, Let $A = (\mu_A, \nu_A)$ be an $(\in, \in)$-intuitionistic fuzzy prime ideal of X. Let $x, y \in X$ and $\mu_A(x \wedge y) = t$ where $t \in [0, 1]$, then
$\mu_A(x \wedge y) \geq t \Rightarrow (x \wedge y)_t \in \mu_A$
$\Rightarrow x_t \in \mu_A$ or $y_t \in \mu_A$
$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t$

$$\Rightarrow \max\{\mu_A(x), \mu_A(y)\} \geq t = \mu_A(x \wedge y) \tag{3.3}$$

Again let $x, y \in X$ such that $\nu_A(x \wedge y) = s$ where $s \in [0, 1]$, then
$\nu_A(x \wedge y) \leq s$
$\Rightarrow \nu_A(x \wedge y) < s + \delta \Rightarrow (x \wedge y)_{s+\delta} \overline{\in} \nu_A$ where δ is arbitrary small
$\Rightarrow x_{s+\delta} \overline{\in} \nu_A$ or $y_{s+\delta} \overline{\in} \nu_A$
$\Rightarrow \nu_A(x) < s + \delta$ or $\nu_A(y) < s + \delta$
$\Rightarrow \nu_A(x) \leq s$ or $\nu_A(y) \leq s$ [Since δ is arbitrary]

$$\Rightarrow \min\{\nu_A(x), \nu_A(y)\} \leq s = \nu_A(x \wedge y) \tag{3.4}$$

Therefore, from equations (3.3) and (3.4) $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy prime ideal of X. $\square$

Theorem 3.3. *An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is a (q, q)-intuitionistic fuzzy prime ideal if and only if $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal.*

Proof. Let $A = (\mu_A, \nu_A)$ be a (q, q)-intuitionistic fuzzy ideal of a BCK-algebra X. Let $x, y \in X$ such that
$(x \wedge y)_t \in \mu_A$
$\Rightarrow \mu_A(x \wedge y) \geq t$
$\Rightarrow \mu_A(x \wedge y) + \delta > t$ [where δ is arbitrary small]
$\Rightarrow \mu_A(x \wedge y) + \delta - t + 1 > 1$
$\Rightarrow (x \wedge y)_{\delta-t+1} q \mu_A$.

Since $A = (\mu_A, \nu_A)$ be a (q, q)- intuitionistic fuzzy ideal. Therefore, we have
$x_{\delta-t+1} q \mu_A$ or $y_{\delta-t+1} q \mu_A$
$\Rightarrow \mu_A(x) + \delta - t + 1 > 1$ or $\mu_A(y) + \delta - t + 1 > 1$
$\Rightarrow \mu_A(x) + \delta \geq t$ or $\mu_A(y) + \delta \geq t$
$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t$ [Since δ is arbitrary]
$\Rightarrow x_t \in \mu_A$ or $y_t \in \mu_A$.

Therefore,

$$(x \wedge y)_t \in \mu_A \Rightarrow x_t \in \mu_A \text{ or } y_t \in \mu_A \tag{3.5}$$

Let $x, y \in X$ such that $(x \wedge y)_s \overline{\in} \nu_A$
$\Rightarrow \nu_A(x \wedge y) < s$
$\Rightarrow \nu_A(x \wedge y) \leq s - \delta$
$\Rightarrow \nu_A(x \wedge y) + \delta - s + 1 \leq 1$
$\Rightarrow (x \wedge y)_{1+\delta-s} \overline{q} \nu_A$.

Since $A = (\mu_A, \nu_A)$ be a (q, q)-intuitionistic fuzzy ideal. Therefore, we have
$x_{1+\delta-s} \overline{q} \nu_A$ or $y_{1+\delta-s} \overline{q} \nu_A$
$\Rightarrow \nu_A(x) + \delta - s + 1 \leq 1$ or $\nu_A(y) + \delta - s + 1 \leq 1$
$\Rightarrow \nu_A(x) \leq s - \delta$ or $\nu_A(y) \leq s - \delta$
$\Rightarrow \nu_A(x) < s$ or $\nu_A(y) < s$ [Since δ is arbitrary]
$\Rightarrow x_s \overline{\in} \nu_A$ or $y_s \overline{\in} \nu_A$.

Therefore,

$$(x \wedge y)_s \overline{\in} \nu_A \Rightarrow x_s \overline{\in} \nu_A \text{ or } y_s \overline{\in} \nu_A \tag{3.6}$$

Therefore, from equations (3.5) and (3.6) $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X.

Conversely, assume $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X. Let
$(x \wedge y)_t q \mu_A$
$\Rightarrow \mu_A(x \wedge y) + t > 1$
$\Rightarrow \mu_A(x \wedge y) > 1 - t$
$\Rightarrow \mu_A(x \wedge y) \geq \delta - t + 1 > 1 - t$ [where $\delta > 0$ is arbitrary]
$\Rightarrow (x \wedge y)_{\delta-t+1} \in \mu_A$.

Since $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X. Therefore, we have $x_{\delta-t+1} \in \mu_A$ or $y_{\delta-t+1} \in \mu_A$
$\Rightarrow \mu_A(x) \geq \delta - t + 1 > 1 - t$ or $\mu(y) \geq \delta - t + 1 > 1 - t$
$\Rightarrow \mu_A(x) > 1 - t$ or $\mu_A(y) > 1 - t$
$\Rightarrow \mu_A(x) + t > 1$ or $\mu_A(y) + t > 1$
$\Rightarrow x_t q \mu_A$ or $y_t q \mu_A$.

Therefore,

$$(x \wedge y)_t \overline{q} \mu_A \Rightarrow x_t q \mu_A \text{ or } y_t q \mu_A \tag{3.7}$$

Let $(x \wedge y)_t \overline{q} \nu_A$
$\Rightarrow \nu_A(x \wedge y) + t \leq 1$
$\Rightarrow \nu_A(x \wedge y) \leq 1 - t$
$\Rightarrow \nu_A(x \wedge y) < \delta - t + 1 < 1 - t$ [where $\delta > 0$ is arbitrary]
$\Rightarrow (x \wedge y)_{1+\delta-t} \overline{\in} \nu_A$.

Since $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X. Therefore, we have $x_{1+\delta-t} \overline{\in} \nu_A$ or $y_{1+\delta-t} \overline{\in} \nu_A$
$\Rightarrow \nu_A(x) < 1 + \delta - t < 1 - t$ or $\nu(y) < 1 + \delta - t + 1 < 1 - t$
$\Rightarrow \nu_A(x) \leq 1 - t$ or $\nu_A(y) \leq 1 - t$
$\Rightarrow \nu_A(x) + t \leq 1$ or $\nu_A(y) + t \leq 1$
$\Rightarrow x_t \overline{q} \nu_A$ or $y_t \overline{q} \nu_A$.

Therefore,

$$(x \wedge y)_t \overline{q} \nu_A \Rightarrow x_t \overline{q} \nu_A \text{ or } y_t \overline{q} \nu_A \tag{3.8}$$

Therefore, from equations (3.7) and (3.8) $A = (\mu_A, \nu_A)$ is a (q, q)-intuitionistic fuzzy prime ideal of X. □

Remark 3.4. The notion of intuitionistic fuzzy prime ideal, $(\in, \in)$-intuitionistic fuzzy prime ideal and (q, q)- intuitionistic fuzzy prime ideals are equivalent.

Theorem 3.5. *An $(\in, \in \vee q))$-intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of a BCK-algebra X is an $(\in, \in \vee q))$-intuitionistic fuzzy prime ideal of X iff*

$$\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\} \quad \forall x, y \in X \tag{3.9}$$

$$\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(x \wedge y), 0.5\} \quad \forall x, y \in X \tag{3.10}$$

Proof. First let $A = (\mu_A, \nu_A)$ be an $(\in, \in \vee q))$-intuitionistic fuzzy prime ideal of X. To prove conditions (3.9) and (3.10), assume that (3.9) is not valid, then there exists some $x, y \in X$ such that

$$\max\{\mu_A(x), \mu_A(y)\} < \min\{\mu_A(x \wedge y), 0.5\},$$

choose a real number t such that

$$\max\{\mu_A(x), \mu_A(y)\} < t < \min\{\mu_A(x \wedge y), 0.5\}, \tag{3.11}$$

then $t \in (0, 0.5]$ and $\mu_A(x \wedge y) > t \Rightarrow (x \wedge y)_t \in \mu_A$, and also $\mu_A(x) < t$ or $\mu_A(y) < t$, i.e., $x_t \overline{\in} \mu_A, y_t \overline{\in} \mu_A$.

Also, $\mu_A(x)+t<2t<2\times 0.5=1$ and $\mu_A(y)+t<2t<2\times 0.5=1$ i.e., $x_t\overline{q}\mu_A,\ y_t\overline{q}\mu_A$.

Hence, $x_t\overline{\in \vee q}\mu_A,\ y_t\overline{\in \vee q}\mu_A$ which is a contradict that $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$)-intuitionistic fuzzy prime ideal. Hence, we must have (3.9).

Assume that (3.10) is not valid, then there exists some $x,y\in X$ such that

$$\min\{\nu_A(x),\nu_A(y)\}>\max\{\nu_A(x\wedge y),0.5\}$$

choose a real number t such that

$$\min\{\nu_A(x),\nu_A(y)\}>t>\max\{\nu_A(x\wedge y),0.5\}. \tag{3.12}$$

Then, $t\in(0.5,1]$ and $\nu_A(x\wedge y)<t\Rightarrow(x\wedge y)_t\overline{\in}\nu_A$, and also $\nu_A(x)>t$ or $\nu_A(y)>t$ i.e., $x_t\in\nu_A, y_t\in\nu_A$.

Also, $\nu_A(x)+t>2t>2\times 0.5=1$ and $\nu_A(y)+t>2t>2\times 0.5=1$ i.e., $x_tq\nu_A,\ y_tq\nu_A$. Hence, $x_t\in\vee q\nu_A,\ y_t\in\vee q\nu_A$ which is a contradict that $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$)-intuitionistic fuzzy prime ideal. Hence, we must have (3.10).

Conversely, suppose that conditions (3.9) and (3.10) hold. Let $x,y\in X$ such that $(x\wedge y)_t\in\mu_A$ where $t\in(0,1]$, i.e., $\mu_A(x\wedge y)\geq t$. Now (3.9) $\Rightarrow\max\{\mu_A(x),\mu_A(y)\}\geq\min\{t,0.5\}$. Now we have

Case I: Let $t\leq 0.5$, then
$\max\{\mu_A(x),\mu_A(y)\}\geq t\ \Rightarrow\mu_A(x)\geq t$ or $\mu_A(y)\geq t$
$\Rightarrow x_t\in\mu_A$ or $y_t\in\mu_A$.

Case II: Let $t>0.5$, then
$\max\{\mu_A(x),\mu_A(y)\}\geq 0.5\Rightarrow\mu_A(x)\geq 0.5$ or $\mu_A(y)\geq 0.5$
$\Rightarrow\mu_A(x)+t\geq 0.5+t>0.5+0.5=1$ or $\mu_A(y)+t\geq 0.5+t>0.5+0.5=1$
$\Rightarrow x_tq\mu_A$ or $y_tq\mu_A$.

Combining Case I and Case II, we get $x_t\in\vee q\mu_A$ or $y_t\in\vee q\mu_A$. Hence, $(x\wedge y)_t\in\mu_A\Rightarrow x_t\in\vee q\mu_A$ or $y_t\in\vee q\mu_A$. Again let $(x\wedge y)_t\overline{\in}\nu_A$ where $t\in(0,1]$, i.e., $\nu_A(x\wedge y)\leq t$. Now (3.10) $\Rightarrow\min\{\nu_A(x),\nu_A(y)\}\leq\max\{t,0.5\}$. Now we have the following cases:

Case I: Let $t>0.5$, then $\min\{\nu_A(x),\nu_A(y)\}<t\ \Rightarrow\nu_A(x)<t$ or $\nu_A(y)<t$
$\Rightarrow x_t\overline{\in}\nu_A$ or $y_t\overline{\in}\nu_A$.

Case II: Let $t\leq 0.5$, then $\min\{\nu_A(x),\nu_A(y)\}\leq 0.5\Rightarrow\nu_A(x)\leq 0.5$ or $\nu_A(y)\leq 0.5$
$\Rightarrow\nu_A(x)+t\leq 0.5+t\leq 0.5+0.5=1$ or $\nu_A(y)+t\leq 0.5+t\leq 0.5+0.5=1$
$\Rightarrow x_t\overline{q}\nu_A$ or $y_t\overline{q}\nu_A$.

Combining Case I and Case II, we get $x_t\overline{\in\vee q}\nu_A$ or $y_t\overline{\in\vee q}\nu_A$.

Hence, $(x\wedge y)_t\overline{\in}\nu_A\Rightarrow x_t\overline{\in\vee q}\nu_A$ or $y_t\overline{\in\vee q}\nu_A$.

Hence, $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$-intuitionistic fuzzy prime ideal of X. □

Remark 3.6. Every intuitionistic fuzzy prime ideal is an $(\in,\in\vee q)$-intuitionistic fuzzy prime ideal.

Remark 3.7. Converse of above is not true as seen from the following example.

Example 3.8. Consider the BCK-algebra $X=\{0,x,y,z\}$ with the following Cayley tables.

Table 1: Illustration of converse of Remark 3.6

$*$	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

$\wedge$	0	x	y	z
0	0	0	0	0
x	0	x	x	0
y	0	x	y	0
z	0	0	0	z

The intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in X defined by $\mu_A(0) = 0.6, \mu_A(x) = 0.5, \mu_A(y) = 0.7, \mu_A(z) = 0.56$ and $\nu_A(0) = 0.3, \nu_A(x) = 0.35, \nu_A(y) = 0.4, \nu_A(z) = 0.5$ then $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal X by Theorem 3.5, but not an intuitionistic fuzzy prime ideal of X because $\mu_A(x \wedge z) = \mu_A(0) = 0.6 > \max\{\mu_A(x), \mu_A(z)\} = \max\{0.5, 0.56\} = 0.56$ and $\nu_A(x \wedge z) = \nu_A(0) = 0.3 < \min\{\nu_A(x), \nu_A(z)\} = \min\{0.35, 0.0.5\} = 0.35$

Theorem 3.9. *If an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a BCK-algebra X is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X and $\mu_A(x) < 0.5$, $\nu_A(x) > 0.5$ $\forall x, y \in X$ then $A = (\mu_A, \nu_A)$ is also an $(\in, \in)$-intuitionistic fuzzy prime ideal of X.*

Proof. Let $A = (\mu_A, \nu_A)$ be an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X and $\mu_A(x) < 0.5$, $\nu_A(x) > 0.5$ $\forall x, y \in X$. Let$(x \wedge y)_t \in \mu_A \quad \Rightarrow \mu_A(x \wedge y) \geq t$ Therefore, $t \leq \mu_A(x \wedge y) < 0.5$ and also $\mu_A(x) < 0.5,\ \mu_A(y) < 0.5$ and $t < 0.5$ Therefore, $\mu_A(x) + t < 0.5 + 0.5 = 1$ and $\mu_A(y) + t < 0.5 + 0.5 = 1 \Rightarrow x_t \overline{q} \mu_A$ and $y_t \overline{q} \mu_A$.

Again let $(x \wedge y)_t \overline{\in} \nu_A \Rightarrow \nu_A(x \wedge y) \leq t$, Therefore, $t \geq \nu_A(x \wedge y) > 0.5$ and also $\nu_A(x) > 0.5,\ \nu_A(y) > 0.5$ and $t > 0.5$; Therefore, $\nu_A(x)+t > 0.5+0.5 = 1$ and $\nu_A(y)+t > 0.5+0.5 = 1$ $\Rightarrow x_t q \nu_A$ and $y_t q \nu_A$. Hence, from above $(x \wedge y)_t \in \mu_A \Rightarrow x_t \overline{q} \mu_A$ and $y_t \overline{q} \mu_A$, $(x \wedge y)_t \overline{\in} \nu_A \Rightarrow x_t q \nu_A$ and $y_t q \nu_A$.

Since $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal, so we must have $(x \wedge y)_t \in \mu_A \Rightarrow x_t \in \mu_A$ and $y_t \in \mu_A$, $(x \wedge y)_t \overline{\in} \nu_A \Rightarrow x_t \overline{\in} \nu_A$ and $y_t \overline{\in} \nu_A$, i.e., $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-intuitionistic fuzzy prime ideal of X. □

Theorem 3.10. *An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of BCK-algebra X is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X if and only if the set $(\mu_A)_t = \{x \in X | \mu_A(x) \geq t,\ t \in (0, 0.5)\}$ and $(\nu_A)_s = \{x \in X | \nu_A(x) < s,\ s \in (0.5, 1]\}$ are prime ideals of X.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. Let $t \in (0, 0.5]$ and $x \wedge y \in (\mu_A)_t$, Therefore, $\mu_A(x \wedge y) \geq t$. It follows that

$$\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_(x \wedge y), 0.5\} \geq \min\{t, 0.5\} = t,$$

Therefore, $\mu_A(x) \geq t$ or $\mu_(y) \geq t$, that is $x \in (\mu_A)_t$, or $y \in (\mu_A)_t$ Therefore, $(\mu_A)_t$ is a prime ideal of X. Let $s \in (0.5, 1]$ and $x \wedge y \overline{\in} (\nu_A)_s$. Therefore, $\nu_A(x \wedge y) < s$. It follows that

$$\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_(x \wedge y), 0.5\} < \max\{s, 0.5\} = s,$$

Therefore, $\nu_A(x) < s$ or $\nu_A(y) < s$, that is $x \overline{\in} (\nu_A)_s$, or $y \overline{\in} (\nu_A)_s$ Therefore, $(\nu_A)_s$ is a prime ideal of X.

Conversely, let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set and the sets $(\mu_A)_t = \{x \in X | \mu_A(x) \geq t,\ \ t \in (0, 0.5)\}$ and $(\nu_A)_s = \{x \in X | \nu_A(x) < s,\ \ s \in (0.5, 1]\}$ are prime ideals of X. To prove $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X.

Suppose $A = (\mu_A, \nu_A)$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X, then there exists some $a, b \in X$ such that at least one of conditions $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge b), 0.5\}$ and $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge b), 0.5\}$ hold.

Suppose $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge b), 0.5\}$ holds. Then choose t such that

$$\max\{\mu_A(a), \mu_A(b)\} < t < \min\{\mu_A(a \wedge b), 0.5\} \tag{3.13}$$

Inequation (3.13) implies $t < \mu_A(a \wedge b)$ i.e. $a \wedge b \in (\mu_A)_t$. Since $(\mu_A)_t$ is an ideal, it follows that $a \in (\mu_A)_t$ or $b \in (\mu_A)_t$ i.e., $\mu_A(a) > t$ or $\mu_A(b) > t$, which contradicts (3.13). Therefore, we must have $\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_(x \wedge y), 0.5\}$.

Again, if $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge b), 0.5\}$ holds, then choose s such that

$$\min\{\nu_A(a), \nu_A(b)\} > s > \max\{\nu_A(a \wedge b), 0.5\} \tag{3.14}$$

Inequation (3.14) implies $s > \mu_A(a \wedge b)$ i.e., $a \wedge b \in (\nu_A)_s$. Since $(\nu_A)_s$ is an ideal, it follows that $a \in (\nu_A)_s$ or $b \in (\nu_A)_s$ i.e. $\nu_A(a) < s$ or $\nu_A(b) < s$ which contradicts (3.14). Therefore, we must have $\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(x \wedge b), 0.5\}$.

Consequently $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. □

Theorem 3.11. *Let A be a non-empty subset of a BCK-algebra X. Consider the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in X defined by*

$$\mu_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{otherwise} \end{cases} \quad \nu_A(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ 1 & \text{otherwise} \end{cases}$$

Then A is a prime ideal of X iff $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X.

Proof. Let A be a prime ideal of X, then $(\mu_A)_t = \{x \in X | \mu_A(x) \geq t\}\ \forall\ t \in (0, 0.5] = A$, and $(\nu_A)_s = \{x \in X | \nu_A(x) \leq s\}\ \forall\ s \in (0.5, 1] = A$, which is a prime ideal. Hence, by Theorem (3.10) $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X.

Conversely, assume that $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X.

Let $x \wedge y \in A$, then $\mu_A(x \wedge y) = 1$.

$\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\} = \min\{1, 0.5\} = 0.5$

$\Rightarrow \max\{\mu_A(x), \mu_A(y)\} \geq 0.5$

$\Rightarrow \mu_A(x) \geq 0.5$ or $\mu_A(y)\} \geq 0.5$

$\Rightarrow \mu_A(x) = 1$ or $\mu_A(y)\} = 1$

$\Rightarrow x \in A$ or $y \in A$.

Again, if $x \wedge y \in A$, then $\nu_A(x \wedge y) = 0$

$\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(x \wedge y), 0.5\} = \max\{0, 0.5\} = 0.5$

$\Rightarrow \min\{\nu_A(x), \nu_A(y)\} \leq 0.5$

$\Rightarrow \nu_A(x) \leq 0.5$ or $\nu_A(y)\} \leq 0.5$

$\Rightarrow \nu_A(x) = 0$ or $\nu_A(y)\} = 0$
$\Rightarrow x \in A$ or $y \in A$.

Therefore, $x \wedge y \in A \Rightarrow x \in A$ or $y \in A$. Hence, A is a prime ideal of X. □

Theorem 3.12. *Let A be a prime ideal of X, then there exists an* $(\in, \in \vee q)$*-intuitionistic fuzzy prime ideal* $A = (\mu_A, \nu_A)$ *of X, such that* $(\mu_A)_t = (\nu_A)_s = S$ *for every* $t \in (0, 0.5]$ *and* $s \in (0.5, 1]$.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ p & \text{otherwise} \end{cases} \quad \nu_A(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ q & \text{otherwise} \end{cases}$$

for all $x \in X$, where $p < t \in (0, 0.5]$, and $q > s \in (0.5, 1]$, $(\mu_A)_t = \{x \in X | \mu_A(x) \geq t > p\} = A$ and $(\nu_A)_s = \{x \in X | \nu_A(x) < s < q\} = A$. Hence, $(\mu_A)_t = (\nu_A)_s = S$ is a prime ideal. Now if $A = (\mu_A, \nu_A)$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. Then there exists some $a, b \in X$ such that at least one of $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge b), 0.5\}$ and $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge b), 0.5\}$ hold.

Suppose that $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge b), 0.5\}$ holds, then choose $t \in (0, 0.5)$ such that

$$\max\{\mu_A(a), \mu_A(b)\} < t < \min\{\mu_A(a \wedge b), 0.5\}. \tag{3.15}$$

Therefore, $\mu_A(a \wedge b) \geq t$ for all $a, b \in X \Rightarrow a \wedge b \in (\mu_A)_t = A$, is a prime ideal.

Therefore, $a \in A$ or $b \in A \Rightarrow \mu_A(a) = 1$ or $\mu_A(b) = 1$, which contradicts (3.15). Hence, we must have $\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\}$.

Again, if $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge b), 0.5\}$ holds, then choose $s \in (0.5, 1]$ such that

$$\min\{\nu_A(a), \nu_A(b)\} > s > \max\{\nu_A(a \wedge b), 0.5\}. \tag{3.16}$$

Therefore, $\nu_A(a \wedge b) \leq s$ for all $a, b \in X \Rightarrow a \wedge b \in (\nu_A)_s = A$, is a prime ideal. Therefore, $a \in A$ or $b \in A \Rightarrow \nu_A(a) = 0$ or $\nu_A(b) = 0$ which contradicts (3.16) Hence, we must have $\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(x \wedge y), 0.5\}$. Hence, $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. □

Definition 3.13. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in BCK-algebra X and $t \in (0\ 1]$, let

$$(\mu_A)_t = \{x \in X | x_t \in \mu_A\} = \{x \in X | \mu_A(x) \geq t\},$$

$$\langle \mu_A \rangle_t = \{x \in X | x_t q \mu_A\} = \{x \in X | \mu_A(x) + t > 1\},$$

$$[\mu_A]_t = \{x \in X | x_t \in \vee q \mu_A\} = \{x \in X | \mu_A(x) \geq t \,\text{or}\, \mu_A(x) + t > 1\}.$$

Here $(\mu_A)_t$ is called t-level set of μ_A, $\langle \mu_A \rangle_t$ is called q-level set of μ_A and $[\mu_A]_t$ is called $(\in \vee q)$-level set of μ_A. Clearly,

$$[\mu_A]_t = \langle \mu_A \rangle_t \cup (\mu_A)_t$$

$$(\nu_A)_t = \{x \in X | x_t \overline{\in} \nu_A\} = \{x \in X | \nu_A(x) < t\}$$

$$< \nu_A >_t = \{x \in X | x_t \bar{q} \nu_A\} = \{x \in X | \nu_A(x) + t \leq 1\}$$
$$[\nu_A]_t = \{x \in X | x_t \overline{\in \vee q} \nu_A\} = \{x \in X | \nu_A(x) < t \,\text{or}\, \nu_A(x) + t \leq 1\}.$$

Here $(\nu_A)_t$ is called t-level set of ν_A, $\langle \nu_A \rangle_t$ is called q-level set of ν_A and $[\nu_A]_t$ is called $(\in \vee q)$-level set of ν_A. Clearly, $[\nu_A]_t = \langle \nu_A \rangle_t \cup (\nu_A)_t$.

Theorem 3.14. *Let $A = (\mu_A, \nu_A)$ be a intuitionistic fuzzy set in BCK-algebra X. Then $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X iff $[\mu_A]_t$ and $[\nu_A]_t$ are prime ideals of X for all $t \in (0\ 1]$. We call $[\mu_A]_t$ and $[\nu_A]_t$ as $(\in \vee q)$-level prime ideals of A.*

Proof. Assume that $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. To prove $[\mu_A]_t$ and $[\nu_A]_t$ are prime ideals of X. Let $x \wedge y \in [\mu_A]_t$ for $t \in (0\ 1]$ then $(x \wedge y)_t \in \vee q \mu_A$ then $\mu_A(x \wedge y) \geq t$ or $\mu_A(x \wedge y) + t > 1$. (Since $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal.) Therefore, $\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\}$. Now we have the following cases:

Case I: $\mu_A(x \wedge y) > t$
$\max\{\mu_A(x), \mu_A(y)\} \geq \min\{t, 0.5\}$
Subcase I: $t > 0.5$
$\max\{\mu_A(x), \mu_A(y)\} \geq 0.5$
$\Rightarrow \mu_A(x) \geq 0.5$ or $\mu_A(y) \geq 0.5$
$\Rightarrow \mu_A(x) + t > 0.5 + 0.5 = 1$ or $\mu_A(y) + t > 0.5 + 0.5 = 1$
$\Rightarrow x_t q \mu_A$ or $y_t q \mu_A$
Subcase II: $t \leq 0.5$
$\max\{\mu_A(x), \mu_A(y)\} \geq t$
$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t$
$\Rightarrow x_t \in \mu_A$ or $y_t \in \mu_A$
Hence, $(x \wedge y)_t \in \vee q \mu_A \Rightarrow x_t \in \vee q \mu_A$ or $y_t \in \vee q \mu_A$.
i.e., $(x \wedge y)_t \in [\mu_A]_t \Rightarrow x_t \in [\mu_A]_t$ or $y_t \in [\mu_A]_t$
Case II: $\mu_A(x \wedge y) + t > 1$
$\max\{\mu_A(x), \mu_A(y)\} \geq \min\{1 - t, 0.5\}$
Subcase I: $t \leq 0.5$
$\max\{\mu_A(x), \mu_A(y)\} \geq 0.5 \geq t$
$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t$
$\Rightarrow x_t \in \mu_A$ or $y_t \in \mu_A$
Subcase II: $t > 0.5$
$\max\{\mu_A(x), \mu_A(y)\} \geq 1 - t$
$\Rightarrow \mu_A(x) \geq 1 - t$ or $\mu_A(y) \geq 1 - t$
$\Rightarrow \mu_A(x) + t \geq 1$ or $\mu_A(y) + t \geq 1$
$\Rightarrow x_t q \mu_A$ or $y_t q \mu_A$.
Hence, $(x \wedge y)_t \in \vee q \mu_A \Rightarrow x_t \in \vee q \mu_A$ or $y_t \in \vee q \mu_A$, i.e., $(x \wedge y)_t \in [\mu_A]_t \Rightarrow x_t \in [\mu_A]_t$ or $y_t \in [\mu_A]_t$.

Similarly, we can prove $(x \wedge y)_t \overline{\in} [\nu_A]_t \Rightarrow x_t \overline{\in} [\nu_A]_t$ or $y_t \overline{\in} [\nu_A]_t$, i.e., $[\mu_A]_t, [\nu_A]_t$ both are prime ideals of X.

Conversely, let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X and $t \in (0\ 1]$ such that $[\mu_A]_t$ and $[\nu_A]_t$ is a prime ideal of X. To prove $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. If $A = (\mu_A, \nu_A)$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X, then there exist some $a, b \in X$ such that at least one of $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge y), 0.5\}$ and $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge y), 0.5\}$ hold. Suppose $\max\{\mu_A(a), \mu_A(b)\} < \min\{\mu_A(a \wedge y), 0.5\}$ holds. Choose t such that

$$\max\{\mu_A(a), \mu_A(b)\} < t < \min\{\mu_A(a \wedge b), 0.5\} \tag{3.17}$$

$\max\{\mu_A(a), \mu_A(b)\} < t < \min\{\mu_A(a \wedge b), 0.5\}$ then $\mu_A(a \wedge b) > t \Rightarrow a \wedge b \in (\mu_A)_t \subseteq [\mu_A]_t$ which is a prime ideal $\Rightarrow a \in [\mu_A]_t$ or $b \in [\mu_A]_t \Rightarrow \mu_A(a) \geq t$ or $\mu_A(a) + t > 1$ or $\mu_A(b) \geq t$ or $\mu_A(b) + t > 1$, which contradicts (3.17). Hence, we must have $\max\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(x \wedge y), 0.5\}$, Again, if $\min\{\nu_A(a), \nu_A(b)\} > \max\{\nu_A(a \wedge y), 0.5\}$ holds. Proceeding as above again we get a contradiction. Hence, we must have $\min\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(x \wedge y), 0.5\}$.

Therefore, $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. □

Theorem 3.15. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two $(\in, \in \vee q)$-intuitionistic fuzzy prime ideals of X. Then $A \cup B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X.*

Proof. Here $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ both are $(\in, \in \vee q)$-intuitionistic fuzzy prime ideals of X. Therefore, $\forall\, x, y \in X$.

$$\begin{aligned} \max\{\mu_A(x), \mu_A(y)\} &\geq \min\{\mu_A(x \wedge y), 0.5\}, \\ \min\{\nu_A(x), \nu_A(y)\} &\leq \max\{\nu_A(x \wedge y), 0.5\}, \\ \max\{\mu_B(x), \mu_B(y)\} &\geq \min\{\mu_B(x \wedge y), 0.5\}, \\ \min\{\nu_B(x), \nu_B(y)\} &\leq \max\{\nu_B(x \wedge y), 0.5\}, \end{aligned} \tag{3.18}$$

We have $(A \cup B)(x) = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle | x \in X\}$, where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

To prove $A \cup B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. It is enough to show that $\forall\, x, y \in X$.

$$\max\{(\mu_A \cup \mu_B(x), (\mu_A \cup \mu_B)(y)\} \geq \min\{(\mu_A \cup \mu_B)(x \wedge y), 0.5\} \tag{3.19}$$

$$\min\{(\nu_A \cup \nu_B(x), (\nu_A \cup \nu_B)(y)\} \leq \max\{(\nu_A \cup \nu_B)(x \wedge y), 0.5\} \tag{3.20}$$

Now $\max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\}$
$= \max\{\max\{\mu_A(x), \mu_B(x)\}, \max\{\mu_A(y), \mu_B(y)\}\}$
$= \max\{\max\{\mu_A(x), \mu_A(y)\}, \max\{\mu_B(x), \mu_B(y)\}\}$
$\geq \max\{\min\{\mu_A(x \wedge y), 0.5\}, \min\{\mu_B(x \wedge y), 0.5\}\}$ by (3.18)

$$\Rightarrow \max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\} \geq \max\{\min\{\mu_A(x \wedge y), 0.5\}, \min\{\mu_B(x \wedge y), 0.5\}\} \tag{3.21}$$

Now we have the following cases:
Case I: $\mu_A(x \wedge y) \leq 0.5$ and $\mu_B(x \wedge y) \leq 0.5$, then

$$(3.21) \Rightarrow \max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\} \geq \max\{\mu_A(x \wedge y), \mu_B(x \wedge y)\}$$
$$\geq (\mu_A \cup \mu_B)(x \wedge y)$$
$$= \min\{(\mu_A \cup \mu_B)(x \wedge y), 0.5\}$$

Case II: $\mu_A(x \wedge y) \leq 0.5$ and $\mu_B(x \wedge y) > 0.5$, then

$$(3.21) \Rightarrow \max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\} \geq \max\{\mu_A(x \wedge y), 0.5\} = 0.5$$
$$= \min\{\max\{\mu_A(x \wedge y), \mu_B(x \wedge y)\}, 0.5\}$$
$$= \min\{(\mu_A \cup \mu_B)(x \wedge y), 0.5\}$$

Case III: $\mu_A(x \wedge y) > 0.5$ and $\mu_B(x \wedge y) \leq 0.5$

$$(3.21) \Rightarrow \max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\} \geq \max\{0.5, \mu_B(x \wedge y)\} = 0.5$$
$$= \min\{\max\{\mu_A(x \wedge y), \mu_B(x \wedge y)\}, 0.5\}$$
$$= \min\{(\mu_A \cup \mu_B)(x \wedge y), 0.5\}$$

Case IV: $\mu_A(x \wedge y) > 0.5$ and $\mu_B(x \wedge y) > 0.5$, then

$$(3.21) \Rightarrow \max\{(\mu_A \cup \mu_B)(x), (\mu_A \cup \mu_B)(y)\} \geq \max\{0.5, 0.5\} = 0.5$$
$$= \min\{\max\{\mu_A(x \wedge y), \mu_B(x \wedge y)\}, 0.5\}$$
$$= \min\{(\mu_A \cup \mu_B)(x \wedge y), 0.5\}$$

Hence, from above, (3.19) holds $\forall\, x, y \in X$.

Similarly, we can show (3.20) holds $\forall\, x, y \in X$. Hence, $(A \cup B)$ is an $(\in, \in \vee q)$-intuitionistic fuzzy prime ideal of X. □

Theorem 3.16. *Let* $\{A_i = (\mu_{A_i}, \nu_{A_i}) | i = 1, 2, ...\}$ *be a family of* $(\in, \in \vee q)$*-intuitionistic fuzzy prime ideals of X, then* $\mu = \cup\{A_i : i = 1, 2, ...\}$ *is an* $(\in, \in \vee q)$*-intuitionistic fuzzy prime ideal of X, where* $\cup A_i(x) = \{\langle x, \min(\mu_{A_i}(x) : i = 1, 2, 3, ...), \max(\mu_{A_i}(x) : i = 1, 2, 3, ...)\rangle | x \in X\}$.

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