Conditional probability on IF-events

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Abstract

Probability on collections of IF-sets can be considered as a generalization of the classical probability theory on σ-algebras of sets. The aim of this contribution is to formulate the version of the conditional probability on IF-events and show its properties. The paper is based on the idea for Łukasiewicz implication, but now there are a lot of different implications in the theory of IF-sets.

Keywords: IF-event, conditional probability.

1 Introduction

The theory of Intuitionistic Fuzzy Sets (IF-sets) was introduced by ATANASSOV. We recall that an IF-set is a couple of functions (μ, ν) with values in the unit interval, such that $\mu + \nu \leq 1$.

We consider a **Łukasiewicz tribe** with product (denoted by $\mathcal{T}$),which is a non empty set of functions $f : \Omega \to [0, 1]$ satisfying the following conditions:

(i) if $f \in \mathcal{T}$ then $1 - f \in \mathcal{T}$,

(ii) if $f, g \in \mathcal{T}$ then $f \oplus g \in \mathcal{T}$,

(iii) if $f_n \in \mathcal{T}(n = 1, 2, ...), f_n \nearrow f$ then $f \in \mathcal{T}$,

(iv) if $f, g \in \mathcal{T}$ then $f.g \in \mathcal{T}$.

The well-known Łukasiewicz operations $\oplus, \odot$ on $\mathcal{T}$ are given by

$$f \oplus g = \min(f + g, 1) = (f + g) \wedge 1, \ \ f \odot g = \max(f + g - 1, 0) = (f + g - 1) \vee 0.$$

Denote by $\mathcal{F}$ the family of IF-events: $\mathcal{F} = \{(f,g); f,g \in \mathcal{T}, f+g \leq 1\}$ together with the operations $\oplus, \odot$:

$$(f_1,g_1) \oplus (f_2,g_2) = (f_1 \oplus f_2, g_1 \odot g_2) = ((f_1+f_2) \wedge 1, (g_1+g_2-1) \vee 0),$$

$$(f_1,g_1) \odot (f_2,g_2) = (f_1 \odot f_2, g_1 \oplus g_2) = ((f_1+f_2-1) \vee 0, (g_1+g_2) \wedge 1),$$
$$\neg(f,g) = (1-f, 1-g).$$

We define an order on $\mathcal{F}$ by $(f_1,g_1) \leq (f_2,g_2) \iff f_1 \leq f_2$ and $g_1 \geq g_2$, and recall that
$$(f_n,g_n) \nearrow (f,g) \iff f_n \nearrow f,\ g_n \searrow g.$$

The **product operation** on $\mathcal{F}$ is a binary operation $\cdot$ defined by

$$(f_1,g_1) \cdot (f_2,g_2) = (f_1.f_2, 1-(1-g_1).(1-g_2)) = (f_1.f_2, g_1+g_2-g_1.g_2).$$

Proposition 1 (Lendelová [6]) *The product operation defined of $\mathcal{F}$ satisfies following conditions:*

(i) $(1,0) \cdot (f,g) = (f,g)$ *for each* $(f,g) \in \mathcal{F}$,

(ii) operation $\cdot$ is commutative and associative,

(iii) if $(f_1,g_1),(f_2,g_2) \in \mathcal{F}$ *and* $(f_1,g_1) \odot (f_2,g_2) = (0,1)$, *then*

$$(f_3,g_3) \cdot ((f_1,g_1) \oplus (f_2,g_2)) = ((f_3,g_3) \cdot (f_1,g_1)) \oplus ((f_3,g_3) \cdot (f_2,g_2))$$

and
$$((f_3,g_3) \cdot (f_1,g_1)) \odot ((f_3,g_3) \cdot (f_2,g_2)) = (0,1)$$
for each $(f_3,g_3) \in \mathcal{M}$,

(iv) if $(f_{1n},g_{1n}),(f_{2n},g_{2n}) \in \mathcal{F}$ *and* $(f_{1n},g_{1n}) \searrow (0,1), (f_{2n},g_{2n}) \searrow (0,1)$, *then* $(f_{1n},g_{1n}) \cdot (f_{2n},g_{2n}) \searrow (0,1)$.

Proof. See [6], *Theorem* 1.

2 Conditional probability

Definition 1 *A **state** on $\mathcal{F}$ is a mapping $m : \mathcal{F} \longrightarrow [0,1]$, which satisfies the following conditions:*

1. $m(1,0) = 1$,
2. *if* $(f_1,g_1) \oplus (f_2,g_2) \leq (1,0)$ *then* $m((f_1,g_1) \oplus (f_2,g_2)) = m(f_1,g_1) + m(f_2,g_2)$,
3. *if* $(f_n,g_n) \nearrow (f,g)$ *then* $m(f_n,g_n) \nearrow m(f,g)$.

Remark The condition *2.* in previous *Definition* can be equivalently written as follows:

$$\text{if } (f_1, g_1) \leq \neg(f_2, g_2) \text{ then } m((f_1, g_1) \oplus (f_2, g_2)) = m(f_1, g_1) + m(f_2, g_2).$$

By induction is easy to prove that

$$\text{if } \bigoplus_{n=1}^{k}(f_n, g_n) \leq (1, 0) \text{ then } m(\bigoplus_{n=1}^{k}(f_n, g_n)) = \sum_{n=1}^{k} m(f_n, g_n).$$

Definition 2 *Denote by $\mathcal{B}(R)$ the Borel σ-algebra. An **observable** on $\mathcal{F}$ is a mapping $y : \mathcal{B}(R) \longrightarrow \mathcal{F}$ satisfying the following conditions:*

1. $y(R) = (1, 0)$,
2. *if* $A \cap B = \emptyset$ *then* $y(A) \odot y(B) = (0, 1)$ *and* $y(A \cup B) = y(A) \oplus y(B)$,
3. *if* $A_n \nearrow A$ *then* $y(A_n) \nearrow y(A)$.

Definition 3 *If m is a state and y is an observable on $\mathcal{F}$, then the **probability distribution of y** is the mapping $m_y : \mathcal{B}(R) \longrightarrow [0, 1]$ given by the formula*

$$m_y(A) = m(y(A)).$$

Theorem 1 *There exists an integrable function $\varphi : R \to R$ such that*

$$\int_a^b \varphi \, d\mu_{\mathcal{F}} = m\left((f, g) \cdot y([a, b))\right)$$

holds for any interval $[a, b)$.

Proof.
Existence of function φ will be proved by with help of embedding $\mathcal{F}$ into MV-algebra. Existence of function $p : R \to R$ satisfying the condition

$$\int_B p \, dm_y = m\left((f, g) \cdot y(B)\right)$$

(for $(f, g) \in \mathcal{M}$, $m : \mathcal{M} \to [0, 1]$, $y : \mathcal{B}(R) \to \mathcal{M}$) was proved in [9].
The considered MV-algebra induced by IF-events was $(\mathcal{M}, \oplus, \odot, \neg, \mathbf{0}, u)$, where

$$\begin{aligned}
\mathcal{M} &= \{(f, g); f, g \in \mathcal{T}, \mathcal{T}\}, \\
(f_1, g_1) \oplus (f_2, g_2) &= ((f_1 + f_2) \wedge 1, (g_1 + g_2 - 1) \vee 0), \\
(f_1, g_1) \odot (f_2, g_2) &= ((f_1 + f_2 - 1) \vee 0, (g_1 + g_2) \wedge 1), \\
\neg(f, g) &= (1 - f, 1 - g), \\
\mathbf{0} &= (0, 1), \\
u &= (1, 0).
\end{aligned}$$

From the facts:

- family $\mathcal{F}$ can be embedded into $\mathcal{M}$,
- there exists one-to-one correspondence between state (probability) on $\mathcal{M}$ and state (probability) on $\mathcal{F}$,

is clear, that the existence of the function φ follows from the existence of version of the conditional probability on MV-algebra $\mathcal{M}$ induced by IF-events.

Definition 4 *Let $(f,g) \in \mathcal{F}$ and $y : \mathcal{B}(R) \to \mathcal{F}$ be an observable. A function $p((f,g)|y) : R \to R$ is a version of the* ***conditional probability*** *of (f,g) with respect to y, if*

$$\int_B p((f,g)|y)dm_y = m((f,g) \cdot y(B))$$

for every $B \in \mathcal{B}(R)$.

Properties of conditional probability (listed in next *Proposition*) follow immediately from the properties of conditional probability defined on MV-algebra $\mathcal{M}$.

Proposition 2 *Let y be an observable, $(f,g) \in \mathcal{F}$. Then $p((f,g)|y)$ has the following properties:*

1. $p((0,1)|y) = 0$, $p((1,0)|y) = 1$ $\quad$ *m_y-almost everywhere,*
2. $0 \leq p((f,g)|y) \leq 1$ $\quad$ *m_y-almost everywhere,*
3. *if* $\widehat{+}_{n=1}^{k}(f_n, g_n) \leq (1,0)$ *then* $p([\widehat{+}_{n=1}^{k}(f_n, g_n)]|y) = \sum_{n=1}^{k} p((f_n, g_n)|y)$ $\quad$ *m_y-almost everywhere,*
4. *if* $(f_n, g_n) \nearrow (f,g)$ *then* $p((f_n, g_n)|y) \nearrow p((f,g)|y)$ $\quad$ *m_y-almost everywhere.*

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