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Solution of intuitionistic fuzzy equation with extended operations

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Abstract

Assuming that $*$ is any operation defined on a product set $X \times Y$ and taking values on a set Z,it can be extended to intuitionistic fuzzy sets by means of the extended form of the Zadeh's extension principle for the intuitionistic fuzzy sets. Given an IFS C of Z, it is here shown how to solve the equation $A * B = C$ (or $A * B \subseteq C$) when an intuitionistic fuzzy subset A of X (or an intuitionistic fuzzy subset B of Y) is given.

1 Introduction

After the introduction of the concept of fuzzy sets by Zadeh [14]. several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of intuitionistic fuzzy set was first published by Atanassov [1, 2, 3, 4], as a generalization of the notion of fuzzy set.

In this paper, using the Atanassovs idea, we establish the intuitionistic fuzzification of the concept of extension principle for intuitionistic fuzzy sets, and we examine the resolution of intuitionistic fuzzy arithmetical operations, i.e. when $*$ stands for *+*, *-*, $\times$ or $\div$, extended to intuitionistic fuzzy numbers.

2 Preliminaries

An intuitionistic fuzzy set (IFS for short) of nonempty set X is defined by Atanassov [1, 2] in the following way.

Definition 1 *An intuitionistic fuzzy set A of a nonempty X is an object having the form*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$$

where the function

$$\mu_A : X \to [0, 1] \text{ and } \nu_A : X \to [0, 1]$$

denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$.
Notation. We will represent as IFS(X) the class all the intuitionistic fuzzy subsets of set X.

3 The extended $*$ operation

Let $*$ be an operation definied on a product set $X \times Y$ and taking values on a set Z. To all (x, y) in $X \times Y$, $*$ associates an element z in Z, which is denoted $x * y$.

The $*$ operation can be extended to intuitionistic fuzzy sets by means of the following extension principle.

Definition 2 *Let A be an intuitionistic fuzzy subset of X and B be an intuitionistic fuzzy subset of Y, then the extension principle allows to define an intuitionistic fuzzy subset $C = A * B$ of Z as follows, in the case of noninteractive variables : $\forall z \in Z$*

$$\mu_{A*B}(z) = \sup_{x*y=z} \mu_A(x) \wedge \mu_B(y) \quad \textit{and} \quad \nu_{A*B}(z) = \inf_{x*y=z} \nu_A(x) \vee \nu_B(y) \tag{1}$$

where, as usual μ_A and ν_A are the membership and the nonmembership functions of IFS A, $\wedge$ and $\vee$ denotes the min and max operations.

Note that, in (1), it is assumed that given z in Z, there exists x in X and y in Y such that $x * y = z$. But if that would not be the case

Given the intuitionistic fuzzy sets A and B, $C = A * B$ is then uniquely determined by definition. The problem we are going to investigate is the inverse one, i.e. from the knowledge of the intuitionistic fuzzy sets C and A (or C and B) find, when it exists, an intuitionistic fuzzy set B (or A) such that $C = A * B$. Note that $*$ is not necessarily a commutative operation. Moreover, even if for a couple (x, y) the result $x * y$ is well defined, that may not be necessarily the case for $y * x$. The sequel of this work will be mostly oriented towards the case where C and A are given (equation $A * \mathcal{X} = C$. The case where C and B are given (equation $\mathcal{X} * \mathcal{B} = \mathcal{C}$ can be easily derived by adapting the formulae. We recall that $*$ is any operation here. A pratical application of our study is of course intuitionistic fuzzy arithmetic when A, B and C are intuitionistic fuzzy numbers, i.e. intuitionistic fuzzy subsets of $\mathbb{R}$, and when $*$ is one of the usual operations $+$, $-$, $\times$, $\div$. For example, $A + B = C$ does not imply $B = C - A$ where : $\forall z \in \mathbb{R}$

$$\mu_{A+B}(z) = \sup_{x+y=z} \mu_A(x) \wedge \mu_B(y) \quad \text{and} \quad \nu_{A+B}(z) = \inf_{x+y=z} \nu_A(x) \vee \nu_B(y) \tag{2}$$

$$\mu_{A-B}(z) = \sup_{x-y=z} \mu_A(x) \wedge \mu_B(y) \quad \text{and} \quad \nu_{A-B}(z) = \inf_{x-y=z} \nu_A(x) \vee \nu_B(y) \tag{3}$$

The extension of $*$ to intuitionistic fuzzy sets is inclusion monotonic. In other terms, for B_1 and B_2 intuitionistic fuzzy subsets of Y.

$$B_1 \subseteq B_2 \;\Rightarrow\; A * B_1 \subseteq A * B_2 \tag{4}$$

for A_1 and A_2 intuitionistic fuzzy subsets of X.

$$A_1 \subseteq A_2 \;\Rightarrow\; A_1 * B \subseteq A_2 * B \tag{5}$$

In fact, the extension of $*$ is distributive over the union of intuitionistic fuzzy sets, i.e.

$$A * (B_1 \cup B_2) = (A * B_1) \cup (A * B_2) \tag{6}$$

$$(A_1 \cup A_2) * B = (A_1 * B) \cup (A_2 * B) \tag{7}$$

moreover,

$$A * (B_1 \cap B_2) = (A * B_1) \cap (A * B_2) \tag{8}$$

$$(A_1 \cap A_2) * B = (A_1 * B) \cap (A_2 * B) \tag{9}$$

A direct consequence of (6) and (7) is that the set of solutions of $A * \mathcal{X} = C$ (or of $\mathcal{X} * B = C$, if non-void, is an upper semi-lattice. But, in view of formulae (6) to (9), the set of solutions of $A * \mathcal{X} \subseteq C$ (or of $\mathcal{X} * B \subseteq C$ is a lattice.

For pratical computations, let us show now one can transform the ewpression of $A * B$ given in (1)

Assume first that B is a (non-fuzzy) singleton identified with its unique element, say b in Y, so that $\mu_B(y) = 1$ if $y = b$ and $\mu_B(y) = 0$ if $y \neq b$. Thus, equation (1) yields : $\forall z \in Z$

$$\mu_{A*b}(z) = \sup_{x*b=z} \mu_A(x) \quad \text{and} \quad \nu_{A*b}(z) = \inf_{x*b=z} \nu_A(x) \tag{10}$$

Equation (10) takes the following simpler form : $\forall z \in Z$

$$\mu_{A*b}(z) = \mu_A(z-b) \quad \text{and} \quad \nu_{A*b}(z) = \nu_A(z-b)$$

Let us return to (1).

$$\begin{aligned} \forall z \in Z,\ \mu_{A*B}(z) &= \sup_{x*y=z} \mu_A(x) \wedge \mu_B(y) \\ &= \sup_{y \in Y} \left(\sup_{x*y=z} \mu_A(x) \wedge \mu_B(y) \right) \\ &= \sup_{y \in Y} \left(\mu_B(y) \wedge \sup_{x*y=z} \mu_A(x) \right) \\ &= \sup_{y \in Y} \mu_B(y) \wedge \mu_{A*y}(z) \end{aligned}$$

and similary, we have

$$\begin{aligned} \forall z \in Z,\ \nu_{A*B}(z) &= \inf_{x*y=z} \nu_A(x) \vee \nu_B(y) \\ &= \inf_{y \in Y} \nu_B(y) \vee \nu_{A*y}(z) \end{aligned}$$

from (10) with y playing the role of b. Hence, (1) is equivalent to

$$\mu_{A*B}(z) = \sup_{y\in Y} \mu_{A*y}(z) \wedge \mu_B(y) \text{ and } \nu_{A*B}(z) = \inf_{y\in Y} \nu_{A*y}(z) \vee \nu_B(y) \tag{11}$$

Analogously, exchanging the roles of A and B, we have

$$\mu_{A*B}(z) = \sup_{x\in X} \mu_A(x) \wedge \mu_{B*x}(z) \text{ and } \nu_{A*B}(z) = \inf_{x\in X} \nu_A(x) \vee \nu_{B*x}(z) \tag{12}$$

Note that with addition of intuitionistic fuzzy numbers, (11) takes the form

$$\sup_{y\in Y} \mu_A(z-y) \wedge \mu_B(y) \quad \text{and} \quad \inf_{y\in Y} \nu_A(z-y) \vee \nu_B(y)$$

4 The α operator

In order to solve the$*$-equation problem on intuitionistic fuzzy sets we need to recall the definition of the α operator which is characteristic of Brouwerian lattices. That α operator has proved to be useful in the resolution of composite fuzzy relation equations [11, 12] and we study here a particular composite, with a constraint expressed by the $*$ operation.

Definition 3 *Given a and b in $[0,1]$, $a\alpha b$ is defined as the greatest element x in $[0,1]$ such that $a \wedge x \leq b$, i.e.*

$$a\alpha b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \tag{13}$$

Here are some properties of the α operator that will be used in the sequel. We recall that, as usual, $\vee$ denotes the *max* operation.
For all a, b in $[0,1]$ and for all family $(b_i)_i \in I$ of elements of $[0,1]$, we have

$$a \wedge (a\alpha b) \leq b \tag{14}$$

$$a\alpha\left(\sup_{i\in I} b_i\right) \geq a\alpha b \quad \forall j \in I \tag{15}$$

$$a\alpha(a \wedge b) \geq b \tag{16}$$

According to (13), properties (14) and (16) are directly verified. To check (15), it suffices to denote $c = \sup_{i\in I,\ i\neq j} b_i$, and to show that $a\alpha(c \vee b_i) \geq a\alpha b_j$.

Definition 4 *Given $A \in IFS\,(X)$, $C \in IFS\,(Z)$ and $* : X \times Y \to Z$, we define $\widetilde{*}$: $IFS\,(X) \times IFS\,(Y) \to IFS\,(Z)$ as follows : $\forall y \in Y$*

$$\mu_{C\widetilde{*}A}(y) = \inf_{x*y=z} \mu_A(x)\alpha\mu_C(z) \quad \text{and} \quad \nu_{C\widetilde{*}A}(y) = \sup_{x*y=z} \nu_A(x)\alpha\nu_C(z) \tag{17}$$

5 Resolution of $*$ equations on IFS

Theorem 1 *For every pair of intuitionistic fuzzy sets A of X and C of Z, and for $* : X \times Y \to Z$, we have*

$$A * (C \widetilde{*} A) \subseteq C \tag{18}$$

In order terms, $C \widetilde{*} A$ is a particular solution to $A * \mathcal{X} \subseteq C$.

Proof . Let $U = A * (C \widetilde{*} A)$ and let $z \in Z$. Then

$$\begin{aligned}
\mu_U(z) &= \sup_{x*y=z} \mu_A(x) \wedge \mu_{C*A}(y) \\
\mu_U(z) &= \sup_{x*y=z} \mu_A(x) \wedge \inf_{x'*y=z'} \mu_A(x')\alpha\mu_C(z') \\
\mu_U(z) &\leq \sup_{x*y=z} \mu_A(x) \wedge [\mu_A(x)\alpha\mu_C(z)] \\
\mu_U(z) &\leq \sup_{x*y=z} \mu_C(z) \\
\mu_U(z) &\leq \mu_C(z)
\end{aligned}$$

and

$$\begin{aligned}
\nu_U(z) &= \inf_{x*y=z} \nu_A(x) \vee \nu_{C*A}(y) \\
\nu_U(z) &= \inf_{x*y=z} \nu_A(x) \vee \sup_{x'*y=z'} \nu_A(x')\alpha\nu_C(z') \\
\nu_U(z) &\geq \inf_{x*y=z} \nu_A(x) \vee [\nu_A(x)\alpha\nu_C(z)] \\
\nu_U(z) &\geq \inf_{x*y=z} \nu_C(z) \\
\nu_U(z) &\geq \nu_C(z)
\end{aligned}$$

Theorem 2 *For every pair of intuitionistic fuzzy sets $A \in IFS(X)$ and $B \in IFS(Y)$, and for $* : X \times Y \to Z$, we have*

$$B \subseteq (A * B) \widetilde{*} A \tag{19}$$

Note that when $A * B = C$, we have $B \subseteq C \widetilde{*} A$

Proof . Let $V = (A * B) \widetilde{*} A$ and let $y \in Y$.

$$\begin{aligned}
\mu_V(z) &= \inf_{x*y=z} \mu_A(x)\alpha\mu_{A*B}(z) \\
\mu_V(z) &= \inf_{x*y=z} \mu_A(x)\alpha \sup_{x'*y'=z} \mu_A(x') \wedge \mu_B(y') \\
\mu_V(z) &\geq \inf_{x*y=z} \mu_A(x)\alpha\,[\mu_A(x) \wedge \mu_B(y)] \\
\mu_V(z) &\geq \inf_{x*y=z} \mu_B(y) \\
\mu_V(z) &\geq \mu_B(y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_V(z) &= \sup_{x*y=z} \nu_A(x)\alpha\nu_{A*B}(z) \\
\nu_V(z) &= \sup_{x*y=z} \nu_A(x)\alpha \inf_{x'*y'=z} \nu_A(x') \vee \nu_B(y') \\
\nu_V(z) &\leq \sup_{x*y=z} \nu_A(x)\alpha \left[\nu_A(x) \vee \nu_B(y)\right] \\
\nu_V(z) &\leq \sup_{x*y=z} \nu_B(y) \\
\nu_V(z) &\leq \nu_B(y)
\end{aligned}$$

Corollary 1 *Given $A \in IFS(X)$, $C \in IFS(Z)$ and $* : X \times Y \to Z$, equation $A * \mathcal{X} \subseteq C$ has always a greatest solution given by $C \widetilde{*} A$. Moreover, the set of solutions of $A * \mathcal{X} \subseteq C$ is a lattice.*

Corollary 2 *For $A \in IFS(X)$, $B \in IFS(Y)$ and $C \in IFS(Z)$, we have*

$$A * B \subseteq C \quad \textit{iff} \quad B \subseteq C \widetilde{*} A \tag{20}$$

Theorem 3 *Given an IFSs $A \in IFS(X)$, $C \in IFS(Z)$ and an operation $* : X \times Y \to Z$, the equation*

$$A * \mathcal{X} = C$$

has a solution if, and only if,

$$A * (C \widetilde{*} A) = C$$

Moreover, when $C \widetilde{*} A$ is a solution, then it is the greatest one and the set of solutions is an upper semi-lattice

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