Notes on Intuitionistic Fuzzy Sets
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On the index matrix representation of intuitionistic fuzzy multigraphs

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Abstract: The concept of an intuitionistic fuzzy multigraph is introduced and its interpretation in frames of index matrix is discussed.
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1 Introduction

Exactly 25 years ago, Shannon and Atanassov introduced the concept of an Intuitionistic Fuzzy Graph (IFG, [8]). During the following years, some modifications and extensions of IFGs were generated and Index Matrix (IM; see [1,4]) interpretations of the standard graphs and IFGs were constructed (see [4]).

Only now, after introducing the idea of three-dimensional IMs (see [4, 9]), we have the possibility of an IM interpretation of the concept of a multigraph. It is well-known that it is a graph which is permitted to have multiple (or parallel) arcs.

First, in Section 2, we give short remarks on IMs and Intuitionistic Fuzzy IMs (IFIMs, see [4]).

In Section 3, we introduce the concept of an IF MultiGraph (IFMG) as an extension of the multigraph.

2 Short remarks on standard and intuitionistic fuzzy index matrices

The concept of IM was introduced in [1] and discussed in more details in [4].

Let I be a fixed set of indices and $\mathcal{R}$ be the set of all real numbers. By IM with index sets K and L $(K, L \subset I)$, we mean the object,

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, \dots, k_m\},\ L = \{l_1, l_2, \dots, l_n\}$, and $a_{k_i,l_j} \in \mathcal{R}$ for $1 \le i \le m$, and $1 \le j \le n$.

On the basis of this definition, an extended object – the Intuitionistic Fuzzy IM (IFIM) – was introduced in the form (see [4]):

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1} \rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2} \rangle & \dots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n} \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$: $\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1]$.

Many operations, relations and operators are defined over IFIM (see [4]), but for the needs of the further discussion we give only the following one.

For the IFIMs $A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P, Q, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, some operations that are analogous to the usual matrix operation of addition are defined, e.g. Addition-$(\max, \min)$, Addition-$(\min, \max)$, Addition-(avg). For our needs, we describe only the latest one:

$$A \oplus_{(\text{avg})} B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and} \\ & \quad v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and} \\ & \quad v_w = l_j \in L; \\ \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and} \\ & \quad v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and} \\ & \quad v_w = q_s \in Q; \\ \\ \langle \frac{1}{2}(\mu_{k_i,l_j} + \rho_{p_r,q_s}), \frac{1}{2}(\nu_{k_i,l_j} + \sigma_{p_r,q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \text{ and} \\ & \quad v_w = l_j = q_s \in L \cap Q; \\ \\ \langle 0, 1 \rangle, & \text{otherwise.} \end{cases}.$$

3 Short remarks on intuitionistic fuzzy graphs

In [2–4], different definitions of IFGs are given. Here, we discuss the definition, given in [4].

Let us have the following oriented graph C (Figure1):

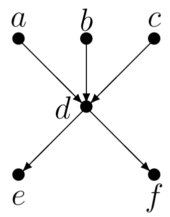

Figure 1. Oriented graph C

For it, we can construct the $(0,1)$-IM which is an adjacency matrix of the graph

$C =$

	a	b	c	d	e	f
a	0	0	0	1	0	0
b	0	0	0	1	0	0
c	0	0	0	1	0	0
d	0	0	0	0	1	1
e	0	0	0	0	0	0
f	0	0	0	0	0	0

Shortly, we denote this matrix by "Adjacency IM" (cf. [7]).

Let $V = \{v_1, v_2, ..., v_n\}$ be a fixed set of vertices and let each vertex x have a degree of existence $\alpha(x)$ and a degree of non-existence $\beta(x)$. Therefore, we can construct the IFS

$$V^* = \{\langle x, \alpha(x), \beta(x) \rangle | x \in V\} = \{\langle v_i, \alpha(v_i), \beta(v_i) \rangle | 1 \le i \le n\},$$

where for each $x \in V$:

$$\alpha(x), \beta(x), \alpha(x) + \beta(x) \in [0, 1].$$

Let H be a set of arcs between the vertices from V. We can again juxtapose to each arc a degree of existence $\mu(x, y)$ and a degree of non-existence $\nu(x, y)$. Therefore, we can construct the new IFS

$$\begin{aligned} H^* &= \{\langle\langle x, y\rangle, \mu(x, y), \nu(x, y)\rangle | x, y \in V\} \\ &= \{\langle\langle v_i, v_j\rangle, \mu(v_i, v_j), \nu(v_i, v_j)\rangle | 1 \leq i, j \leq n\}, \end{aligned}$$

where for each $x, y \in V$:

$$\mu(x, y), \;\; \nu(x, y), \;\; \mu(x, y) + \nu(x, y) \in [0, 1].$$

For the ordinary graph $G = (V, H)$ we can construct the Extended Intuitionistic Fuzzy Graph (EIFG) $G^* = (V^*, H^*)$. It has the following IM-representation:

$$[V^*, V^*, \{\langle \mu(v_i, v_j), \nu(v_i, v_j)\rangle\}]$$

$$= \begin{array}{c|ccc} & v_1, \langle\alpha(v_1), \beta(v_1)\rangle & \dots & v_n, \langle\alpha(v_n), \beta(v_n)\rangle \\ \hline v_1, \langle\alpha(v_1), \beta(v_1)\rangle & \langle\mu_{v_1,v_1}, \nu_{v_1,v_1}\rangle & \dots & \langle\mu_{v_1,v_n}, \nu_{v_1,v_n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ v_i, \langle\alpha(v_i), \beta(v_i)\rangle & \langle\mu_{v_i,v_1}, \nu_{v_i,v_1}\rangle & \dots & \langle\mu_{v_i,v_n}, \nu_{v_i,v_n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ v_n, \langle\alpha(v_n), \beta(v_n)\rangle & \langle\mu_{v_n,v_1}, \nu_{v_n,v_1}\rangle & \dots & \langle\mu_{v_n,v_n}, \nu_{v_n,v_n}\rangle \end{array},$$

where for every $1 \leq i \leq n, 1 \leq j \leq n$: $\mu_{v_i,v_j}, \nu_{v_i,v_j} \in [0, 1], \mu_{v_i,v_j} + \nu_{v_i,v_j} \in [0, 1]$, $\alpha(v_i)$, $\beta(v_i) \in [0, 1], \alpha(v_i) + \beta(v_i) \in [0, 1]$.

4 Main results

Let us consider a standard (within the present research) oriented multigraph – e.g., the following multigraph G (Figure 2).

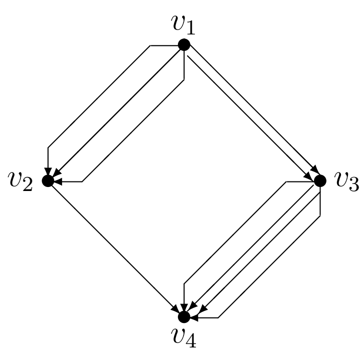

Figure 2. Oriented multigraph G

For this graph we can construct the IM

$$G^* = [\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}, \{\alpha_{i,j}\}] = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 3 & 2 & 0 \\ v_2 & 0 & 0 & 0 & 1 \\ v_3 & 0 & 0 & 0 & 4 \\ v_4 & 0 & 0 & 0 & 0 \end{array},$$

or, as it is discussed in [4], it can have the reduced form

$$G^* = \begin{array}{c|ccc} & v_2 & v_3 & v_4 \\ \hline v_1 & 3 & 2 & 0 \\ v_2 & 0 & 0 & 1 \\ v_3 & 0 & 0 & 4 \end{array}.$$

Let us juxtapose to its vertices the degrees of existence and of non-existence in the form of the IFS $\{\langle v_i, \alpha(v_i), \beta(v_i)\rangle | 1 \le i \le n\}$. Now, let each arc between vertices v_i and v_j be also numbered by an Intuitionistic Fuzzy Pair (IFP, see [6]) $\langle \mu_{v_i,v_j,k}, \nu_{v_i,v_j,k}\rangle$, where $1 \le k \le s_{i,j}$, $\mu_{v_i,v_j,k}, \nu_{v_i,v_j,k}, \mu_{v_i,v_j,k} + \nu_{v_i,v_j,k} \in [0, 1]$, and $s_{i,j}$ is the number of the arcs between vertices v_i and v_j. Therefore, we obtain an (oriented) IFMG.

For example, the considered above multigraph G can have the form from Figure 3.

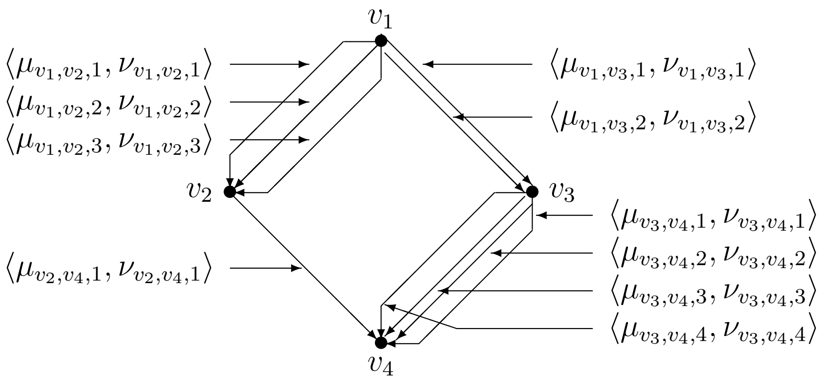

Figure 3. Resultant oriented intuitionistic fuzzy multigraph

Therefore, we can determine the number

$$g(G) = \max_{1 \le i \le j \le n} s_{i,j}$$

and we can construct $g(G)$ in number IMs, that for the considered example have the forms

$$X_1 = \begin{array}{c|cccc} 1 & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & \langle 0,1\rangle & \langle \mu_{v_1,v_2,1}, \nu_{v_1,v_2,1}\rangle & \langle \mu_{v_1,v_3,1}, \nu_{v_1,v_3,1}\rangle & \langle 0,1\rangle \\ v_2 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle \mu_{v_2,v_4,1}, \nu_{v_2,v_4,1}\rangle \\ v_3 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle \mu_{v_3,v_4,1}, \nu_{v_3,v_4,1}\rangle \\ v_4 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{array},$$

$$X_2 = \begin{array}{c|cccc} 2 & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & \langle 0,1\rangle & \langle \mu_{v_1,v_2,2}, \nu_{v_1,v_2,2}\rangle & \langle \mu_{v_1,v_3,2}, \nu_{v_1,v_3,2}\rangle & \langle 0,1\rangle \\ v_2 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ v_3 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle \mu_{v_3,v_4,2}, \nu_{v_3,v_4,2}\rangle \\ v_4 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{array},$$

$$X_3 = \begin{array}{c|cccc} 3 & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & \langle 0,1\rangle & \langle \mu_{v_1,v_2,3}, \nu_{v_1,v_2,3}\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ v_2 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ v_3 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle \mu_{v_3,v_4,3}, \nu_{v_3,v_4,3}\rangle \\ v_4 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{array},$$

$$X_4 = \begin{array}{c|cccc} 4 & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ v_2 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ v_3 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle \mu_{v_3,v_4,4}, \nu_{v_3,v_4,4}\rangle \\ v_4 & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{array}.$$

Hence, to the above multigraph G we can juxtapose a three-dimensional IM (3D-IM) G^* with the form from Figure 4.

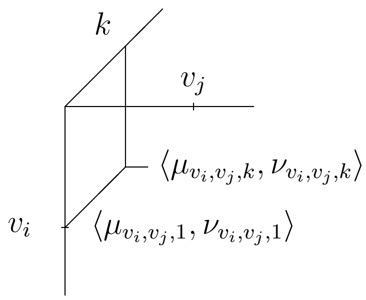

Figure 4. The 3D-IM G^* corresponding to the above multigraph G

It is important to note that the construction we have discussed is applicable in the case of non-oriented graphs without changes.

Now, we can introduce some possible conditions for correctness of the IFMGs, for each of the vertices v_i and v_j that are connected with the (oriented) arc with initial vertex v_i and end vertex v_j:

- **strong condition**:

$$\alpha_i \geq \max_{1 \leq k \leq s_{i,j}} \mu_{v_i,v_j,k} \quad \text{and} \quad \beta_i \leq \min_{1 \leq k \leq s_{i,j}} \nu_{v_i,v_j,k},$$

$$\alpha_j \leq \min_{1 \leq k \leq s_{i,j}} \mu_{v_i,v_j,k} \quad \text{and} \quad \beta_j \geq \max_{1 \leq k \leq s_{i,j}} \nu_{v_i,v_j,k}.$$

- **weak condition**:

$$\alpha_i \geq \min_{1\leq k\leq s_{i,j}} \mu_{v_i,v_j,k} \quad \text{and} \quad \beta_i \leq \max_{1\leq k\leq s_{i,j}} \nu_{v_i,v_j,k},$$

$$\alpha_j \leq \max_{1\leq k\leq s_{i,j}} \mu_{v_i,v_j,k} \quad \text{and} \quad \beta_j \geq \min_{1\leq k\leq s_{i,j}} \nu_{v_i,v_j,k}.$$

- **average condition**:

$$\alpha_i \geq \frac{1}{s_{i,j}}\sum_{k=1}^{s_{i,j}} \mu_{v_i,v_j,k} \geq \alpha_j \quad \text{and} \quad \beta_i \leq \frac{1}{s_{i,j}}\sum_{k=1}^{s_{i,j}} \nu_{v_i,v_j,k} \leq \beta_j.$$

Finally, to each pair (v_i, v_j), we can juxtapose the (standard, 2D) IM

$$Y_{(v_i,v_j)} = \begin{array}{c|cccc} & 1 & 2 & \cdots & s_{i,j} \\ \hline (i,j) & \langle \mu_{v_i,v_j,1}, \nu_{v_i,v_j,1}\rangle & \langle \mu_{v_i,v_j,2}, \nu_{v_i,v_j,2}\rangle & \cdots & \langle \mu_{v_i,v_j,s_{i,j}}, \nu_{v_i,v_j,s_{i,j}}\rangle \end{array}.$$

Now, using the new IM we can transform the IFMG to different types of ordinary IFGs. For this reason, we construct an IFG with the same vertices as in the original IFMG, but with only one (in the discussed case) oriented arc between each pair of neighbouring vertices from the IFMG. Each one of these arcs has a pair degrees of existence and non-existence, that can be determined by the following (at least three) different ways, that determine the form of the IFG:

- **strong IFG**:

$$\langle \max_{1\leq k\leq s_{i,j}} \mu_{v_i,v_j,k}, \min_{1\leq k\leq s_{i,j}} \nu_{v_i,v_j,k}\rangle;$$

- **average IFG**:

$$\langle \frac{1}{s_{i,j}} \sum_{1\leq k\leq s_{i,j}} \mu_{v_i,v_j,k}, \frac{1}{s_{i,j}} \sum_{1\leq k\leq s_{i,j}} \nu_{v_i,v_j,k}\rangle;$$

- **weak IFG**:

$$\langle \min_{1\leq k\leq s_{i,j}} \mu_{v_i,v_j,k}, \max_{1\leq k\leq s_{i,j}} \nu_{v_i,v_j,k}\rangle.$$

5 Conclusion

The IFMGs introduced in this paper can be utilized in a variety of different applications, such as cognitive maps, OLAP-structures with intuitionistic fuzzy estimations of their elements, intercriteria analysis, and others. In future, the IFMGs will be extended to interval-valued IFMGs using ideas from [5, 9].

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References

[1] Atanassov, K. (1987). Generalized index matrices, *Comptes rendus de l'Academie Bulgare des Sciences*, 40 (11), 15–18.

[2] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer, Heidelberg.

[3] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.

[4] Atanassov, K. (2014). *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham.

[5] Atanassov, K. (2020). *Interval-Valued Intuitionistic Fuzzy Sets*, Springer, Cham.

[6] Atanassov, K., Szmidt, E. & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19 (3), 1–13.

[7] Harary, F. (1994). *Graph Theory*. Reading, MA: Addison-Wesley.

[8] Shannon, A., & Atanassov K. (1994). A first step to a theory of the intuitionistic fuzzy graphs, *Proceedings of the First Workshop on Fuzzy Based Expert Systems (D. Lakov, Ed.), Sofia, Sept. 28-30, 1994*, 59–61.

[9] Traneva, V., & Tranev, S. (2017). *Index Matrices as a Tool for Managerial Decision Making*, Avangard Publishing House, Sofia (in Bulgarian).