Notes on Intuitionistic Fuzzy Sets
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On the intuitionistic fuzzy implication $\rightarrow_{191}$

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In memory of my friend Ivan Georgiev

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Abstract: A new intutionistic fuzzy implication, $\rightarrow_{191}$, is constructed. Some of its basic properties are studied.
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1 Introduction

In [3], 185 intuituinistic fuzzy implications are defined. Five other intuituinistic fuzzy implications are introduced in [4–10]. Now a new intuituinistic fuzzy implication will be given.

In some definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Let everywhere intuitionistic fuzzy truth values of variables x and y be

$$x = \langle a, b \rangle, \quad y = \langle c, d \rangle.$$

In [3], for the variables x and y operation "conjunction" (&) is defined by:

$$V(x \& y) = \langle \min(a, c), \max(b, d) \rangle.$$

The pair $\langle a, b \rangle$ is: a tautology if and only if (iff) $a = 1$ and $b = 0$, and is an intuitionistic fuzzy tautology (IFT) iff $a \geq b$.

2 Main results

Let us define

$$x \to_{191} y = \langle a, b \rangle \to \langle c, d \rangle = \langle \overline{\mathrm{sg}}(a - c)\, \overline{\mathrm{sg}}(d - b), \mathrm{sg}(a - c)\, \mathrm{sg}(d - b) \rangle.$$

For brevity, below we will write $\to$ instead of $\to_{191}$.

First, we will show that the definition of the new implication is correct. Let $a, b, c, d \in [0, 1]$ such that $a + b \leq 1$ and $c + d \leq 1$. Then

$$0 \leq \overline{\mathrm{sg}}(a - c)\, \overline{\mathrm{sg}}(d - b) \leq 1,$$

$$0 \leq \mathrm{sg}(a - c)\, \mathrm{sg}(d - b) \leq 1,$$

and if

$$X \equiv \overline{\mathrm{sg}}(a - c)\, \overline{\mathrm{sg}}(d - b) + \mathrm{sg}(a - c)\, \mathrm{sg}(d - b),$$

then, we obtain sequentially. If $a > c$, then $\mathrm{sg}(a - c) = 1$ and $\overline{\mathrm{sg}}(a - c) = 0$, i.e.,

$$X = 0 + \mathrm{sg}(d - b) \leq 1.$$

If $a \leq c$, then, $\mathrm{sg}(a - c) = 0$ and $\overline{\mathrm{sg}}(a - c) = 1$, i.e., $X = \overline{\mathrm{sg}}(d - b) \leq 1$.

Therefore, the definition of the new implication is correct. It generates the following negation.

$$\neg \langle a, b \rangle = \langle a, b \rangle \to \langle 0, 1 \rangle = \langle \overline{\mathrm{sg}}(a - 0)\, \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a - 0)\, \mathrm{sg}(1 - b) \rangle$$

$$= \langle \overline{\mathrm{sg}}(a)\, \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a)\, \mathrm{sg}(1 - b) \rangle = \langle \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a) \rangle$$

$$= \begin{cases} \langle 1, 0 \rangle, & \text{if } a = 0 \text{ and } b = 1 \\ \langle 0, 0 \rangle, & \text{if } a = 0 \text{ and } b < 1 \\ & \text{or } a > 0 \text{ and } b = 1 \\ \langle 0, 1 \rangle, & \text{if } a > 0 \text{ and } b < 1 \end{cases}$$

Theorem 1. *The new implication* $\to_{191}$*:*

(a) satisfies $x \to x$ *as a tautology;*

(b) satisfies $x \to \neg\neg x$ *as a tautology;*

(c) does not satisfy $\neg\neg x \to x$ *even as an IFT.*

Proof. First, we see that

$$\overline{\mathrm{sg}}(1-\mathrm{sg}(a)) = \begin{cases} \text{if } a=0: \overline{\mathrm{sg}}(1)=0 \\ \text{if } a>0: \overline{\mathrm{sg}}(0)=1 \end{cases} = \mathrm{sg}(a),$$

$$\mathrm{sg}(\overline{\mathrm{sg}}(1-b)) = \begin{cases} \text{if } b=1: \mathrm{sg}(1)=1 \\ \text{if } b<1: \mathrm{sg}(0)=0 \end{cases} = \overline{\mathrm{sg}}(1-b),$$

$$\overline{\mathrm{sg}}(a-\mathrm{sg}(a)) = \begin{cases} \text{if } a=0: \overline{\mathrm{sg}}(0)=1 \\ \text{if } a>0: \overline{\mathrm{sg}}(0)=1 \end{cases} = 1,$$

$$\mathrm{sg}(a-\mathrm{sg}(a)) = \begin{cases} \text{if } a=0: \mathrm{sg}(0)=0 \\ \text{if } a>0: \mathrm{sg}(a-1)=0 \end{cases} = 0,$$

$$\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(1-b)-b) = \begin{cases} \text{if } b=1: \overline{\mathrm{sg}}(0)=1 \\ \text{if } b<1: \overline{\mathrm{sg}}(0)=1 \end{cases} = 1,$$

$$\overline{\mathrm{sg}}(\mathrm{sg}(a)-a) = \begin{cases} \text{if } a=0: \overline{\mathrm{sg}}(0)=1 \\ \text{if } a=0: \overline{\mathrm{sg}}(0)=1 \\ \text{if } 0<a<1: \overline{\mathrm{sg}}(1)=0 \end{cases},$$

$$\overline{\mathrm{sg}}(b-\overline{\mathrm{sg}}(1-b)) = \begin{cases} \text{if } b=0: \overline{\mathrm{sg}}(0)=1 \\ \text{if } b=1: \overline{\mathrm{sg}}(0)=1 \\ \text{if } 0<b<1: \overline{\mathrm{sg}}(1)=0 \end{cases},$$

Then

$$\neg\neg\langle a,b\rangle = \neg\langle\overline{\mathrm{sg}}(1-b),\mathrm{sg}(a)\rangle = \langle\overline{\mathrm{sg}}(1-\mathrm{sg}(a)),\mathrm{sg}(\overline{\mathrm{sg}}(1-b))\rangle$$

For (a) we obtain:

$$x\to x = \langle a,b\rangle \to \langle a,b\rangle = \langle\overline{\mathrm{sg}}(0)\,\overline{\mathrm{sg}}(0),\mathrm{sg}(0)\,\mathrm{sg}(0)\rangle = \langle 1,0\rangle.$$

For (b) we obtain:

$$x\to\neg\neg x = \langle a,b\rangle \to \neg\neg\langle a,b\rangle$$

$$= \langle a,b\rangle \to \langle\overline{\mathrm{sg}}(1-\mathrm{sg}(a)),\mathrm{sg}(\overline{\mathrm{sg}}(1-b))\rangle$$

$$= \langle\overline{\mathrm{sg}}(a-\mathrm{sg}(a))\,\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(1-b)-b),\mathrm{sg}(a-\mathrm{sg}(a))\,\mathrm{sg}(\overline{\mathrm{sg}}(1-b)-b)\rangle$$

$$= \langle\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(1-b)-b),0.\,\mathrm{sg}(\overline{\mathrm{sg}}(1-b)-b)\rangle = \langle 1,0\rangle.$$

For (c) we see directly that

$$\neg\neg x\to x = \neg\neg\langle a,b\rangle \to \langle a,b\rangle$$

$$= \langle\overline{\mathrm{sg}}(1-\mathrm{sg}(a)),\mathrm{sg}(\overline{\mathrm{sg}}(1-b))\rangle \to \langle a,b\rangle$$

$$= \langle\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(1-\mathrm{sg}(a))-a)\,\overline{\mathrm{sg}}(b-\mathrm{sg}(\overline{\mathrm{sg}}(1-b))),\mathrm{sg}(\overline{\mathrm{sg}}(1-\mathrm{sg}(a))-a)\,\mathrm{sg}(b-\mathrm{sg}(\overline{\mathrm{sg}}(1-b)))\rangle$$

$$= \langle\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(1-\mathrm{sg}(a))-a)\,\overline{\mathrm{sg}}(b-\mathrm{sg}(\overline{\mathrm{sg}}(1-b))),\mathrm{sg}(\overline{\mathrm{sg}}(1-\mathrm{sg}(a))-a)\,\mathrm{sg}(b-\mathrm{sg}(\overline{\mathrm{sg}}(1-b)))\rangle$$

(for $0 < a, b < 1$)

$$= \langle 0, 1 \rangle,$$

i.e., it is not an IFT. $\square$

Theorem 2. *The new implication $\rightarrow_{191}$:*

(a) satisfies Modus Ponens in tautological sense,

(b) does not satisfy Modus Ponens in the IFT-sense,

(c) satisfies for every two variables x and y,

$$(x \& (x \rightarrow y)) \rightarrow y$$

in the IFT-sense.

Proof. (a) Let $\langle a, b \rangle$ be a tautology, i.e., $a = 1$ and $b = 0$ and let $\langle a, b \rangle \rightarrow \langle c, d \rangle$ be a tautology, i.e., $\langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b) = 1$ and $\text{sg}(a-c)\,\text{sg}(d-b) = 0$. Then $1 = \overline{\text{sg}}(1-c)\,\overline{\text{sg}}(d-0) = \overline{\text{sg}}(1-c)\,\overline{\text{sg}}(d)$, i.e. $1 = \overline{\text{sg}}(1-c)$ and hence $c = 1$ and $d = 0$. Therefore $\langle c, d \rangle$ is a tautology.
(b) Let $\langle a, b \rangle = \langle 0, 0 \rangle$, i.e., an IFT. Then, $\langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), \text{sg}(a-c)\,\text{sg}(d-b) \rangle$ will be an IFT, e.g., for $\langle c, d \rangle = \langle 0.1, 0.2 \rangle$, but the last pair is not an IFT.
(c) we have sequentially:

$$V((x \& (x \rightarrow y)) \rightarrow y)$$

$$= (\langle a, b \rangle \& \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), \text{sg}(a-c)\,\text{sg}(d-b) \rangle) \rightarrow \langle c, d \rangle$$

$$= \langle \min(a, \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)), \max(b, \text{sg}(a-c)\,\text{sg}(d-b)) \rangle \rightarrow \langle c, d \rangle$$

$$= \langle \overline{\text{sg}}(\min(a, \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)) - c)\,\overline{\text{sg}}(d - \max(b, \text{sg}(a-c)\,\text{sg}(d-b)),$$

$$\text{sg}(\min(a, \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)) - c)\,\text{sg}(d - \max(b, \text{sg}(a-c)\,\text{sg}(d-b))) \rangle.$$

Let

$$X \equiv \overline{\text{sg}}(\min(a, \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)) - c)\,\overline{\text{sg}}(d - \max(b, \text{sg}(a-c)\,\text{sg}(d-b))$$

$$- \text{sg}(\min(a, \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)) - c)\,\text{sg}(d - \max(b, \text{sg}(a-c)\,\text{sg}(d-b))).$$

If $a > c$, then

$$X = \overline{\text{sg}}(\min(a, 0) - c)\,\overline{\text{sg}}(d - \max(b, \text{sg}(d-b)) - \text{sg}(\min(a, 0) - c)\,\text{sg}(d - \max(b, \text{sg}(d-b))$$

$$= \overline{\text{sg}}(-c)\,\overline{\text{sg}}(d - \max(b, \text{sg}(d-b)) - \text{sg}(-c)\,\text{sg}(d - \max(b, \text{sg}(d-b)))$$

$$= \overline{\text{sg}}(d - \max(b, \text{sg}(d-b)) \geq 0.$$

If $a \leq c$, then

$$X = \overline{\text{sg}}(\min(a, \overline{\text{sg}}(d-b)) - c)\,\overline{\text{sg}}(d - \max(b, 0)) - \text{sg}(\min(a, \overline{\text{sg}}(d-b)) - c)\,\text{sg}(d - \max(b, 0))$$

$$= \overline{\text{sg}}(\min(a, \overline{\text{sg}}(d-b)) - c)\,\overline{\text{sg}}(d-b)) - \text{sg}(\min(a, \overline{\text{sg}}(d-b)) - c)\,\text{sg}(d-b).$$

If $d > b$, then

$$X = \overline{\text{sg}}(\min(a,0) - c).0 - \text{sg}(\min(a,1) - c) = 0 - \text{sg}(a - c) = 0.$$

If $d \le b$, then

$$X = \overline{\text{sg}}(\min(a,1) - c) - \text{sg}(\min(a,1) - c).0 = \overline{\text{sg}}(a - c) = 1,$$

i.e., the expression is an IFT. □

Following [3], let us define for x

$$\Box x = \langle a, 1 - a \rangle.$$

Theorem 3. *The new implication $\to_{191}$ satisfies the formula*

$$\Box(x \to y) \to (\Box x \to \Box y) \qquad (*)$$

as a tautology.

Proof. We obtain sequentially

$$\begin{aligned}
&\Box(x \to y) \to (\Box x \to \Box y)\\
&= \Box(\langle a, b\rangle \to \langle c, d\rangle) \to (\Box \langle a, b\rangle \to \Box \langle c, d\rangle)\\
&= \Box(\langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), \text{sg}(a-c)\,\text{sg}(d-b)\rangle) \to (\langle a, 1-a\rangle \to \langle c, 1-c\rangle)\\
&= \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), 1-\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)\rangle \to \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(1-c-1+a), \text{sg}(a-c)\,\text{sg}(1-c-1+a)\rangle\\
&= \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), 1-\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)\rangle \to \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(a-c), \text{sg}(a-c)\,\text{sg}(a-c)\rangle
\end{aligned}$$

(because $\overline{\text{sg}}(p)\,\overline{\text{sg}}(p) = \overline{\text{sg}}(p)$ and $\text{sg}(p)\,\text{sg}(p) = \text{sg}(p)$ for each $p \in [0,1]$)

$$\begin{aligned}
&= \langle \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b), 1-\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)\rangle \to \langle \overline{\text{sg}}(a-c), \text{sg}(a-c)\rangle\\
&= \langle \overline{\text{sg}}(\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b) - \overline{\text{sg}}(a-c))\,\overline{\text{sg}}(\text{sg}(a-c) - 1 + \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)),\\
&\quad \text{sg}(\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b) - \overline{\text{sg}}(a-c))\,\text{sg}(\text{sg}(a-c) - 1 + \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b))\rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X = &\langle \overline{\text{sg}}(\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b) - \overline{\text{sg}}(a-c))\,\overline{\text{sg}}(\text{sg}(a-c) - 1 + \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b)),\\
&\text{sg}(\overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b) - \overline{\text{sg}}(a-c))\,\text{sg}(\text{sg}(a-c) - 1 + \overline{\text{sg}}(a-c)\,\overline{\text{sg}}(d-b))\rangle.
\end{aligned}$$

If $a > c$, then

$$\begin{aligned}
x = &\langle \overline{\text{sg}}(0.\,\overline{\text{sg}}(d-b) - 0)\,\overline{\text{sg}}(1 - 1 + 0.\,\overline{\text{sg}}(d-b)), \text{sg}(0.\,\overline{\text{sg}}(d-b) - 0))\,\text{sg}(1 - 1 + \overline{\text{sg}}(d-b))\rangle\\
&= \langle \overline{\text{sg}}(0)\,\overline{\text{sg}}(0), \text{sg}(0)\,\text{sg}(\overline{\text{sg}}(d-b))\rangle = \langle 1, 0\rangle.
\end{aligned}$$

If $a \le c$, then

$$X = \langle \overline{\text{sg}}(\overline{\text{sg}}(d-b) - 1)\,\overline{\text{sg}}(-1 + \overline{\text{sg}}(d-b)), \text{sg}(\overline{\text{sg}}(d-b) - 1)\,\text{sg}(-1 + \overline{\text{sg}}(d-b))\rangle$$

(because $\overline{\text{sg}}(d-b) \le 1$)

$$X = \langle \overline{\text{sg}}(0), \text{sg}(\overline{\text{sg}}(d-b) - 1)\rangle = \langle 1, 0\rangle.$$

Therefore formula $(*)$ is a tautology. Hence, it is an IFT, too. □

3 Conclusion

In this paper we have introduced the 191-st intuitionistic fuzzy implication, and have shown that it has intuitionistic but not classical behaviour. Several of its properties were formulated and proved. In a future research, new properties of the new implications will be studied.

Open Problem: In [2] six Cartesian products are defined (the last of them was published in [1]). For which index j the j-th Cartesian product satisfies equality

$$(A \to_{191} B) \times_j C = (A \times_j C) \to_{191} (B \times_j C)$$

for every three IFSs A, B, C?

References

[1] Andonov, V. (2008). On some properties of one Cartesian product over intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 14 (1), 12–19.

[2] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.

[3] Atanassov K. (2017). *Intuitionistic Fuzzy Logics.* Springer, Cham, 2017.

[4] Atanassov, K., Angelova, N., Szmidt, E, & Kacprzyk, J. (2016). Properties of the intuitionistic fuzzy implication $\to_{186}$. *Notes on Intuitionistic Fuzzy Sets*, 22 (4), 6–12.

[5] Atanassov, K., Ribagin, S., Doukovska, L. & Atanassova, V. (2017). Intuitionistic fuzzy implication $\to_{190}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (4), 79–83.

[6] Atanassov, K., Szmidt, E. & Kacprzyk, J. (2016). New Fodor's type of intuitionistic fuzzy implication and negation. *Notes on Intuitionistic Fuzzy Sets*, 22 (3), 1–8.

[7] Atanassov, K., Szmidt, E. & Kacprzyk, J. (2017). Intuitionistic fuzzy implication $\to_{188}$. Notes on Intuitionistic Fuzzy Sets, 23 (1), 6–13.

[8] Atanassov, K., Szmidt, E. & Kacprzyk, J. (2017). Intuitionistic fuzzy implication $\to_{187}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (2), 37–43.

[9] Atanassov, K., Szmidt, E. & Angelova, N. (2017). Properties of the intuitionistic fuzzy implication $\to_{187}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (3), 3–8.

[10] Atanassova, L. (2017). Intuitionistic fuzzy implication $\to_{189}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (1), 14–20.