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ON PROBABILITY AND INDEPENDENCE IN INTUITIONISTIC FUZZY SET THEORY

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ABSTRACT. The present article includes a synthetic approach to the concept of probability in intuitionistic fuzzy theory [2,3] and gives some remarks and thoughts concerning the conception of total probability and Bayes' theorem.

1. PRELIMINARY REMARKS

According to K. Atanasov, we have [1] :

Definition 1. By a intuitionistic fuzzy set A in the universum $X \neq \emptyset$ we mean the structure

(1.1) $$A = \{(x, \mu_A(x), \nu_A(x) : x \in X\}$$

where the functions $\mu_A, \nu_A : X \to [0,1]$ satisfy the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

and describe, respectively, the degree of the membership and the nonmembership of an element x to A.

The family of all intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

In particular, the intuitionistic fuzzy set $\{(x,1,0) : x \in X\}$ is called a intuitionistic fuzzy space and $\{(x,0,1) : x \in X\}$ the intuitionistic fuzzy empty set, denoted by $\emptyset$.

Moreover, for $A, B \in IFS(X)$, we have [1] :

(1.2) $$A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x),$$

(1.3) $$A = B \Leftrightarrow A \subset B \text{ and } B \subset A,$$

(1.4) $$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},$$

and, simultaneously

$$\nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\},$$

(1.5) $$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\},$$

and, simultaneously

$$\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\},$$

(1.6) $$\mu_{A'}(x) = \nu_A(x) \text{ and } \nu_{A'}(x) = \mu_A(x),$$

(1.7) $$\mu_{A\odot B}(x) = \mu_A(x) \cdot \mu_B(x) \text{ and}$$

$$\nu_{A\odot B}(x) = \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x) \text{ for all } x \in X$$

For the above operations, true is

Theorem 1. *The operations $\cup$ and $\cap$ are commutative, associative, distributive, idempotent and satisfy de Morgan's laws.*

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2. THE GENERAL CONCEPT OF PROBABILITY IN $IFS(X)$

Let $(E, \mathcal{M}, P)$ be a probability space in the usual sense. Let $IFS(E)$ denote the family of all intuitionistic fuzzy sets on E whose membership functions μ and nonmembership functions ν are Borel measurable.

Definition 2. We say that A, B from $IFS(E)$ are equivalent, and denote $A = B$ when $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for almost all $x \in E$.

The relation " $=$ " is an equivalent relation in $IFS(E)$ and , as such, divides the set $IFS(E)$ into abstract classes whose elements are all intuitionistic fuzzy sets equivalent to the given representative. The quotient space $IFS(E)/=$ consisting of all intuitionistic fuzzy sets with Borel measurable functions μ and ν is denoted by $I\mathcal{M}(E)$.

Definition 3. Each intuitionistic fuzzy set A from the family $I\mathcal{M}(E)$ is called a intuitionistic fuzzy event, and the number

$$\widetilde{P}(A) = \int_E \frac{\mu_A(x) + 1 - \nu_A(x)}{2} P(dx) \tag{2.1}$$

its probability.

Let us notice that formula (2.1) in the case when $\nu_A(x) = 1 - \mu_A(x)$, i.e. in the case of the Zadeh fuzziness, reduces to the probability of a fuzzy event introduced in [4] by L.A. Zadeh, and when E is finite, formula (2.1) expresses the probability of a intuitionistic fuzzy event proposed in the first paper on intuitionistic fuzzy probability [2].

It is also true

Theorem 2

(2.2) $\widetilde{P}(A) \geq 0$ *for all* $A \in I\mathcal{M}(E)$,

(2.3) $\widetilde{P}(U) = 1$ *where* U *is a intuitionistic fuzzy space,*

(2.4) $A \cap B = \emptyset \Rightarrow \widetilde{P}(A \cup B) = \widetilde{P}(A) + \widetilde{P}(B)$ *for all* $A, B \in I\mathcal{M}(E)$,

with that the operations above are given in (1.3) and (1.4).

We can see that the function $\widetilde{P} : I\mathcal{M}(E) \to [0,1]$ given in Definition 3 satisfies the clasical axioms in the Kolmogorov definition of probability, which justifies its name "probability". It also satisfies other properties, so much characteristic of the Kolmogorov probability. For example [3]:

Theorem 3

(2.5) $\widetilde{P}(\emptyset) = 0$,

(2.6) $\widetilde{P}(A') = 1 - \widetilde{P}(A)$,

(2.7) $A \subset B \Rightarrow \widetilde{P}(A) \leq \widetilde{P}(B)$,

(2.8) $\widetilde{P}(A \cup B) = \widetilde{P}(A) + \widetilde{P}(B) - \widetilde{P}(A \cap B)$ *for any* $A, B \in I\mathcal{M}(E)$.

The proofs of Theorems 2 and 3 follow from the properties of the Lebesque- Stiltjes integral and of the assumption that P is a probability measure in the common sense.

As in the usual probability theory, we shall call the intuitionistic fuzzy empty set an impossible event, and the intuitionistic fuzzy space a sure event.

3. The concept of independence

Let us start with example. Let us consider the double cast with a symmetric die. Let A denote the event that the number occures on the first cube will be less then 3, B - that the number on the second die is greater then 4. These events are, of course, independent according to the probability theory in the usual sense. We would like to introduce the independence concept for intuitionistic fuzzy events in such a way that the events from the above example are independent, too. So, let us assume that the defining functions of the intuitionistic fuzzy events A and B are shown in Tables 1 and 2 and also 3 and 4.

μ_A	1	...	6
1	1	...	1
2	0.6	...	0.6
3	0.1	...	0.1
4	0	...	0
5	0	...	0
6	0	...	0

Table 1

ν_A	1	...	6
1	0	...	0
2	0.1	...	0.1
3	0.6	...	0.6
4	0.8	...	0.8
5	0.9	...	0.9
6	1	...	1

Table 2

μ_B	1	...	6
1	0	...	0
2	0	...	0
3	0	...	0
4	0.2	...	0.2
5	0.6	...	0.6
6	1	...	1

Table 3

μ_B	1	...	6
1	0.9	...	0.9
2	0.7	...	0.7
3	0.4	...	0.4
4	0.3	...	0.3
5	0.1	...	0.1
6	0	...	0

Table 4

$\mu_{A\cap B}$	1	...	6
1	0	...	0
2	0	...	0
3	0	...	0
4	0	...	0
5	0	...	0
6	0	...	0

Table 5

$\nu_{A\cap B}$	1	...	6
1	0.9	...	0.9
2	0.7	...	0.7
3	0.6	...	0.6
4	0.8	...	0.8
5	0.9	...	0.9
6	1	...	1

Table 6

The numbers given in the Tables 1 – 4 are arbitrary assumed but they fulfil the conditions from Definition 1. The set $A \cap B$ is the defined by the defining functions $\mu_{A\cap B}$ and $\nu_{A\cap b}$ given in Tables 5 and 6 (see 1.4).

And then we have, according to formula (2.1), the probabilities which can easily be verified:

$$\widetilde{P}(A \cap B) = \frac{11}{12}, \quad \widetilde{P}(A) = \frac{49}{120}, \quad \widetilde{P}(B) = \frac{54}{120}.$$

As is clearly seen, $\widetilde{P}(A \cap B) \neq \widetilde{P}(A) \cdot \widetilde{P}(B)$ and this means that the classical definition of independence in the form of $P(A \cap B) = P(A) \cdot P(B)$ does not play its intuitive role. L.A. Zadeh in [4] suggested the use of the algebraic operation $\odot$ instead of the usual intersection $\cap$. According to [4], we may introduce

Definition 4. The events A and B are called independent when

$$\widetilde{P}(A \odot B) = \widetilde{P}(A) \cdot \widetilde{P}(B) \tag{3.1}$$

where the operation $\odot$ is defined as (1.7).

It is not difficult to verify that, in this sense, the events A and B described in the example are independent.

It is also true [3]

Theorem 4. *Any intuitionistic fuzzy event and a intuitionistic fuzzy sure event are independent as well as any intuitionistic fuzzy event and a intuitionistic fuzzy impossible event are independent.*

As an immediate consequence we have the following

Definition 5. By the conditional probability of an event A under the condition B we mean the number

$$\widetilde{P}(A|B) = \frac{\widetilde{P}(A \odot B)}{\widetilde{P}(B)}, \tag{3.2}$$

provided $\widetilde{P}(B) > 0$.

In the case when A and B are independent in the sense of the definition above, $\widetilde{P}(A|B) = \widetilde{P}(A)$, which means that the event B does not have any influence on the event A (similarly as it is in the clasical probability theory).

It is also possible to show [3]

Theorem 5

$$\widetilde{P}(U|C) = 1 \quad \textit{for all } C \in I\mathcal{M}(E) \tag{3.3}$$

and U being a intuitionistic fuzzy sure event,

$$\widetilde{P}(\emptyset|C) = 0, \quad C \in I\mathcal{M}(E), \tag{3.4}$$

$$A \cap B = \emptyset \Rightarrow \widetilde{P}((A \cup B)|C) = \widetilde{P}(A|C) + \widetilde{P}(B|C) \textit{ for all } A, B \in I\mathcal{M}(E) \tag{3.5}$$

$$\widetilde{P}(A|U) = \widetilde{P}(A), \tag{3.6}$$

which only confirms the adopted name of "conditional probability".

4. The total probability theorem. Remarks

Let us note that the automatical transmision of the total probability theorem to the intuitionistic fuzzy event area is not possible. Really, the assumptions of this theorem require that $B_1, B_2, \ldots, B_n$ satisfy the conditions: $P(B_i) > 0$, $B_i \cap B_j = \emptyset$ and $\bigcup_{i=1}^{n} B_i = E$. These assumptions are fulfilled if and only if the event are usual and not intuitionistic fuzzy or even fuzzy in the sense of Zadeh.

Let us assume that there are, for example, two intuitionistic fuzzy events B_1 and B_2, i.e. $B_1 = \{(x, \mu_1(x), \nu_1(x)) : x \in E\}$ and $B_2 = \{(x, \mu_2(x), \nu_2(x)) : x \in E\}$, such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = E$. So, there must be : $\min\{\mu_1(x), \mu_2(x)\} = 0$ and $\max\{\nu_1(x), \nu_2(x)\} = 1$ and, simultaneously, $\max\{\mu_1(x), \mu_2(x)\} = 1$ and $\min\{\nu_1(x), \nu_2(x)\} = 0$. The above conditions are fulfilled only when $\mu_1(x) \in \{0, 1\}$ (which makes $\nu_1(x) \in \{0, 1\}$ interchangeable with $\mu_1(x)$) and $\mu_2(x) \in \{0, 1\}$ (which makes $\nu_2(x) \in \{0, 1\}$ interchangeable with $\mu_2(x)$) under the simultaneous condition $0 \leq \mu(x) + \nu(x) \leq 1$ (in this situation, simply $\mu_1(x) + \nu_1(x) = 1$ and $\mu_2(x) + \nu_2(x) = 1$).

The same reasons make that Bayes' theorem although possible to formulate in intuitionistic fuzzy set theory, comes to the situation where the sets are usual, no fuzzy. How to solve it ? This problem will be discussed in the next papers.

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