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Intuitionistic L-Fuzzy congruence on a ring

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Abstract: In this paper we discuss and study some properties of intuitionistic L-fuzzy equivalence relation on a ring. We introduce the concept of intuitionistic L-fuzzy transitive closure of an intuitionistic L-fuzzy relation on a ring. Further the definition of intuitionistic L-fuzzy congruence relation on a ring is introduced and proved that the set of intuitionistic L-fuzzy congruences forms a modular lattice.
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1 Introduction

The idea of fuzzy sets introduced by L. A. Zadeh [20] is an approach to mathematical representation of vagueness in everyday curriculum. In 1971, A. Rosenfeld [19] initiated the study of applying the notion of fuzzy sets in group theory. Later, K. T. Atanassov [1] in 1983 introduced the notion of intuitionistic fuzzy sets, which is a generalization of fuzzy sets. In 1984 K. T. Atanassov and S. Stoeva [2] introduced intuitionistic L-fuzzy sets which is a generalization of L-fuzzy sets. The foundation laid by K. T. Atanassov, by introducing intuitionistic fuzzy sets, has tremendously inspired researchers of the boarder frame work of fuzzy setting, which further flourished fuzzy mathematics. The idea of intuitionistic fuzzy subgroup initiated by R. Biswas in [7] illustrates the application of intuitionistic fuzzy sets in group theory. Likewise in [6] Banerjee and Basnet introduced Intuitionstic fuzzy subrings and ideals. H. Bustince and P. Burillo [8] introduced the concept of intuitionistic fuzzy relations on a set and studied some properties. In [12, 13, 14, 15] Kul Hur and his colleagues investigated intuitionistic fuzzy equivalence relation on a set and intuitionistic fuzzy congruence on a groupoid (semigroup or lattice). Many researchers have applied the notion of intuitionistic fuzzy sets to the fields of Sociometry, Medical diagnosis, Decision Making, Logic Programming, Artificial Intelligence etc [9, 11, 16, 21].

In this paper we generalize the work of earlier authors in the framework of intuitionistic L-fuzzy sets. We discuss some properties of intuitionistic L-fuzzy equivalence relation on a ring. Further we introduce intuitionistic L-fuzzy transitive closure of an intuitionistic L-fuzzy relation. An intuitionistic L-fuzzy congruence relation on a ring is defined and we prove some results on these. Finally, we establish that the intuitionistic L-fuzzy congruence on a ring forms a modular lattice.

2 Preliminaries

In this section, we recall some basic concepts of intuitionistic L-fuzzy sets and intuitionistic L-fuzzy relation [4, 5, 17, 18]. We introduce intuitionistic L-fuzzy equivalence relation on a ring and state their elementary properties. In this paper, L denotes a complete distributive lattice with maximal element 1 and minimal element 0, respectively. Let R denote a commutative ring with binary operation denoted by "$+$" and "$\cdot$".

Definition 1 ([2]). *Let $(L, \leq)$ be the lattice with an involutive order reversing operation $N : L \to L$. Let X be a non-empty set. An intuitionistic L-fuzzy set (ILFS) A in X is defined as,*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

where $\mu_A : X \to L$ define the membership and $\nu_A : X \to L$ define the non-membership function of every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

Definition 2 ([2]). *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$ be two intuitionistic L-fuzzy sets of X. Then we define*

(i) $A \subseteq B$ *iff for all* $x \in X$, $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$

(ii) $A = B$ *iff for all* $x \in X$, $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$

(iii) $A \cup B = \{\langle x, (\mu_A \vee \mu_B)(x), (\nu_A \wedge \nu_B)(x)\rangle / x \in X\}$

(iv) $A \cap B = \{\langle x, (\mu_A \wedge \mu_B)(x), (\nu_A \vee \nu_B)(x)\rangle / x \in X\}$.

Definition 3 ([2]). *Let $\{A_i\}_{i \in I}$ be an arbitrary family of ILFSs in X where*

$$A_i = \{\langle x, \mu_{A_i}(x), \nu_{A_i}(x)\rangle / x \in X\}, \quad i \in I.$$

Then

(i) $\cap A_i = \{\langle x, \wedge_{i \in I}\mu_{A_i}(x), \vee_{i \in I}\nu_{A_i}(x)\rangle / x \in X\}$

(ii) $\cup A_i = \{\langle x, \vee_{i \in I}\mu_{A_i}(x), \wedge_{i \in I}\nu_{A_i}(x)\rangle / x \in X\}$.

Definition 4. Let $A = \{\langle (x, y), \mu_A(x, y), \nu_A(x, y)\rangle / (x, y) \in R \times R\}$ be an ILFS over $R \times R$. Then A is called an intuitionistic L-fuzzy relation on R (ILFR (R)) if for all $(x, y) \in R \times R$, $\mu_A(x, y) \leq N(\nu_A(x, y))$, where $N : L \to L$, $\mu_A : R \times R \to L$ and $\nu_A : R \times R \to L$.

Definition 5. Let $A \in \text{ILFR}(R)$. Then the inverse of A denoted by A^{-1} is defined as follows: for $x, y \in R$

$$A^{-1}(x,y) = A(y,x).$$

Definition 6. [3] Let $A = \{\langle (x,y), \mu_A(x,y), \nu_A(x,y)\rangle / (x,y) \in R \times R\}$ and $B = \{\langle (x,y), \mu_B(x,y), \nu_B(x,y)\rangle / (x,y) \in R \times R\}$ be ILFR (R). Then the composition $A \circ B$ of A and B is defined as follows: for $x, y \in R$

$$A \circ B = \{\langle (x,y), \mu_{A\circ B}(x,y), \nu_{A\circ B}(x,y)\rangle / (x,y) \in R \times R\}$$

$$\text{where } \mu_{A\circ B}(x,y) = \vee_{z\in R}(\mu_A(x,z) \wedge \mu_B(z,y))$$
$$\text{and } \nu_{A\circ B}(x,y) = \wedge_{z\in R}(\nu_A(x,z) \vee \nu_B(z,y)).$$

Definition 7. Let $A = \{\langle (x,y), \mu_A(x,y), \nu_A(x,y)\rangle / (x,y) \in R \times R\}$ be an ILFR (R). Then A is called an intuitionistic L-fuzzy equivalence relation on R (ILFER (R)) if it satisfies the following conditions:

(i) intuitionistic L-fuzzy reflexive
i.e. $\mu_A(x,x) = 1$, $\nu_A(x,x) = 0$ for all $x \in R$

(ii) intuitionistic L-fuzzy symmetric
i.e. $\mu_A(x,y) = \mu_A(y,x)$ and $\nu_A(x,y) = \nu_A(y,x)$ for all $x, y \in R$
i.e. $A = A^{-1}$

(iii) intuitionistic L-fuzzy transitive
i.e. $\mu_A(x,y) \geq \vee_{z\in R}(\mu_A(x,z) \wedge \mu_A(z,y))$ and $\nu_A(x,y) \leq \wedge_{z\in R}(\nu_A(x,z) \vee \nu_A(z,y))$
i.e. $A \circ A \subseteq A$.

The following results are immediate.

Proposition 1. *Let $P_1, P_2, Q_1, Q_2 \in$ ILFR (R). Then*

(i) $(P_1 \circ P_2) \circ P_3 = P_1 \circ (P_2 \circ P_3)$.

(ii) If $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$ then $P_1 \circ Q_1 \subseteq P_2 \circ Q_2$. In particular if $Q_1 \subseteq Q_2$ then $P_1 \circ Q_1 \subseteq P_1 \circ Q_2$.

(iii) $(P_1^{-1})^{-1} = P_1$.

(iv) $(P_1 \cup P_2)^{-1} = P_1^{-1} \cup P_2^{-1}$.

Proposition 2. *Let $P_1, Q_1 \in$ ILFR (R). If $Q_1 \circ P_1 = P_1 \circ Q_1$ then*

$$(Q_1 \circ P_1) \circ (Q_1 \circ P_1) = (Q_1 \circ Q_1) \circ (P_1 \circ P_1).$$

Proposition 3. *Let $P, Q \in$ ILFR (R). Then*

(i) If P, Q are intuitionistic L-fuzzy symmetric then $P \cup Q$ is intuitionistic L-fuzzy symmetric.

(ii) If $P \subseteq Q$ then $P^{-1} \subseteq Q^{-1}$.

(iii) If $P, Q \in$ ILFER (R) then $P \cap Q$, $P \circ P \in$ ILFER (R).

Proposition 4. *If P is an ILFER (R) then $P \circ P = P$.*

Definition 8. Let $\Delta = \{\langle (x,y), \mu_\Delta(x,y), \nu_\Delta(x,y)\rangle/(x,y) \in R \times R\}$ and

$$\nabla = \{\langle (x,y), \mu_\nabla(x,y), \nu_\nabla(x,y)\rangle/(x,y) \in R \times R\}$$

be two ILFR (R) such that for $(x,y) \in R \times R$,
(i)

$$\mu_\Delta(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

and

$$\nu_\Delta(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

(ii) $\mu_\nabla(x,y) = 1$,
$\nu_\nabla(x,y) = 0$.
Then $\Delta, \nabla \in$ ILFER (R).

Proposition 5. *Let*

$$A_i = \{\langle (x,y), \mu_{A_i}(x,y), \nu_{A_i}(x,y)\rangle/(x,y) \in R \times R, i \in I\}$$

be an ILFER (R). Then

$$A = \{\langle (x,y), \mu_A(x,y), \nu_A(x,y)\rangle/(x,y) \in R \times R\} \text{is an ILFER } (R)$$

where $\mu_A(x,y) = \wedge_{i\in I}\mu_{A_i}(x,y)$, $\quad \nu_A(x,y) = \vee_{i \in I}\nu_{A_i}(x,y)$.

Remark. If $A, B \in$ ILFER (R) then $A \cup B$ need not be ILFER (R).

Proposition 6. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle/(x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle/(x,y) \in R \times R\}$$

be intuitionistic L-fuzzy reflexive. Then, $P \circ Q$ is intuitionistic L-fuzzy reflexive.

Proposition 7. *Let $P, Q \in$ ILFER (R). If $Q \circ P = P \circ Q$, then $P \circ Q$ is an ILFER (R).*

3 Lattice of intuitionistic L-fuzzy equivalence relations

We define an ILFER generated by an intuitionistic L-fuzzy relation and the intuitionistic L-fuzzy transitive closure of an intuitionistic L-fuzzy relation. Here we study some elementary properties of intuitionistic L-fuzzy equivalence relation (ILFER) and we prove that it forms a complete lattice.

Definition 9. Let $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$ be an ILFR (R). Let $\{P_\alpha\}_{\alpha \in I}$ be the family of ILFER (R) containing P. Then $\cap P_\alpha$ containing P is called the ILFER (R) generated by P and denoted by P^e. It is the smallest ILFER (R) containing P.

The following definition is based on a result of S. Kumar De, R. Biswas and A. R. Roy [10].

Definition 10. Let $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$ be an ILFR (R). Then the intuitionistic L-fuzzy transitive closure of P denoted by P^∞ is defined as follows:

$$P^\infty = \cup_{n \in \mathbb{N}} P^n$$

where $P^n = P \circ P \circ \cdots \circ P$, in which P occurs n times. Here

$$P^\infty = \{\langle (x,y), \mu_{P^\infty}(x,y), \nu_{p^\infty}(x,y)\rangle / (x,y) \in R \times R\}$$

where $\mu_{P^\infty}(x,y) = \vee_{n \in \mathbb{N}} \mu_{p^n}(x,y)$ and $\nu_{P^\infty}(x,y) = \wedge_{n \in \mathbb{N}} \nu_{p^n}(x,y)$.

The following results are straightforward.

Proposition 8. *Let $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$ be an ILFR (R). Then, P^∞ is the smallest intuitionistic L-fuzzy transitive relation containing P.*

Proposition 9. *Let $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$ be an intuitionistic L-fuzzy symmetric on R. Then,*

$$P^\infty = \{\langle (x,y), \mu_{P^\infty}(x,y), \nu_{P^\infty}(x,y)\rangle / (x,y) \in R \times R\}$$

is an intuitionistic L-fuzzy symmetric on R.

Proposition 10. *Let $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$*
and $Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$ be the ILFR (R). Then
(i) If $P \subseteq Q$ then $P^\infty \subseteq Q^\infty$.
(ii) If $P \circ Q = Q \circ P$ and $P, Q \in$ ILFER (R) then $(P \circ Q)^\infty = P \circ Q$.

Theorem 1. *If $P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$ is an ILFR (R) then $P^e = [P \cup P^{-1} \cup \Delta]^\infty$.*

Proof. Let $Q = [P \cup P^{-1} \cup \Delta]^\infty$. Clearly, $P \subseteq Q$ and Q is intuitionistic L-fuzzy transitive. Let $x \in R$. Then

$$1 = \mu_\Delta(x,x) \le \mu_Q(x,x)$$

and

$$0 = \nu_\Delta(x,x) \ge \nu_Q(x,x).$$

Thus, $\mu_Q(x,x) = 1$ and $\nu_Q(x,x) = 0$. Hence, Q is intuitionistic L-fuzzy reflexive.

Also $P \cup P^{-1} \cup \Delta = [P \cup P^{-1} \cup \Delta]^{-1}$. Hence, $Q = [P \cup P^{-1} \cup \Delta]^\infty$ is intuitionistic L-fuzzy symmetric. Therefore, $Q \in$ ILFER (R).

Let $S \in$ ILFER (R) such that $P \subseteq S$. Then, $\Delta \subseteq S$ and $P^{-1} \subseteq S$. Hence $P \cup P^{-1} \cup \Delta \subseteq S$. i.e. $[P \cup P^{-1} \cup \Delta]^n \subseteq S^n = S$ for $n \ge 1$. Hence, $Q \subseteq S$.

Therefore, Q is the smallest ILFER (R) containing P. Hence,

$$Q = P^e = [P \cup P^{-1} \cup \Delta]^\infty.$$

□

The following results are straightforward.

Proposition 11. *Let*

$$P = \{\langle (x,y), \mu_p(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFER (R)*. Define* $P \vee Q = (P \cup Q)^{\infty} = \cup_{n \in \mathbb{N}} (P \cup Q)^n$*. Then,* $P \vee Q \in$ *ILFER* (R).

Proposition 12. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFER (R)*. If* $P \circ Q \in$ *ILFER* (R) *then* $(P \circ Q)^{\infty} = P \circ Q$.

The join of two ILFER (R) can also be given as follows.

Theorem 2. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle | (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle | (x,y) \in R \times R\}$$

be ILFER (R).
If $P \circ Q \in$ *ILFER* (R) *then* $P \circ Q = P \vee Q$*, where* $P \vee Q$ *is the least upper bound for* $\{P, Q\}$ *with respect to inclusion.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned} \mu_{P \circ Q}(x,y) &= \vee_{z \in R} (\mu_P(x,z) \wedge \mu_Q(z,y)) \\ &\geq \mu_P(x,y) \wedge \mu_Q(y,y) \\ &= \mu_P(x,y) \end{aligned}$$

and

$$\begin{aligned} \nu_{P \circ Q}(x,y) &= \wedge_{z \in R} (\nu_P(x,z) \vee \nu_Q(z,y)) \\ &\leq \nu_P(x,y) \vee \nu_Q(y,y) \\ &= \nu_P(x,y). \end{aligned}$$

Hence, $P \subseteq P \circ Q$. Similarly $Q \subseteq P \circ Q$. Hence, $P \circ Q$ is an upper bound for $\{P, Q\}$ with respect to '$\subseteq$'.

Let $S \in$ ILFER (R) such that $S \supseteq P$ and $S \supseteq Q$. Let $x, y \in R$. Then,

$$\begin{aligned}\mu_{P\circ Q}(x,y) &= \vee_{z\in R}(\mu_P(x,z) \wedge \mu_Q(z,y)) \\ &\leq \vee_{z\in R}(\mu_S(x,z) \wedge \mu_S(z,y)) \\ &= \mu_{S\circ S}(x,y) \\ &= \mu_S(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\circ Q}(x,y) &= \wedge_{z\in R}(\nu_P(x,z) \vee \nu_Q(z,y)) \\ &\geq \wedge_{z\in R}(\nu_S(x,z) \vee \nu_S(z,y)) \\ &= \nu_{S\circ S}(x,y) \\ &= \nu_S(x,y).\end{aligned}$$

Thus, $P \circ Q \subseteq S$. Hence $P \circ Q$ is the least upper bound for $\{P, Q\}$. Consequently, $P \circ Q = P \vee Q$. □

Proposition 13. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFER (R)*. Then,* $P \vee Q = (P \circ Q)^{\infty}$.

Proof. Clearly for $P, Q \in$ ILFER (R)

$$\begin{aligned}P \vee Q = (P \cup Q)^e &= [(P \cup Q) \cup (P \cup Q)^{-1} \cup \Delta]^{\infty} \\ &= [P \cup Q]^{\infty}.\end{aligned}$$

Also, $P \subseteq P \cup Q$, $Q \subseteq P \cup Q$ and therefore

$$P \circ Q \subseteq (P \cup Q) \circ (P \cup Q) = P \cup Q.$$

Hence, $(P \circ Q)^{\infty} \subseteq (P \cup Q)^{\infty} = P \vee Q$.
Since $P, Q \in$ ILFER (R), $P \subseteq P \circ Q$ and $Q \subseteq P \circ Q$ which implies that

$$P \cup Q \subseteq P \circ Q.$$

Therefore,

$$(P \cup Q)^{\infty} \subseteq (P \circ Q)^{\infty}$$

which implies that

$$P \vee Q \subseteq (P \circ Q)^{\infty}.$$

Hence,

$$P \vee Q = (P \circ Q)^{\infty}.$$

□

The set of ILFER (R) is a poset with respect to "$\subseteq$". Define two operations $\vee$, $\wedge$ on ILFER (R) as follows: for $P, Q \in$ ILFER (R), $P \wedge Q = P \cap Q$ and $P \vee Q = (P \cup Q)^e = (P \circ Q)^\infty$.

Theorem 3. *The set (ILFER (R), $\vee$, $\wedge$) forms a complete lattice with least element Δ and greatest element ∇.*

4 Intuitionistic L-fuzzy congruences

In this section, we define an intuitionistic L-fuzzy congruence on a ring and study its properties.

Definition 11. Let $P = \{\langle(x,y), \mu_P(x,y), \nu_P(x,y)\rangle/(x,y) \in R \times R\}$ be an ILFER (R). Then
(i) P is called an intuitionistic L-fuzzy left congruence on R,
(ILFLC (R))if, for all $a, b, c, d, x \in R$

$$\begin{aligned} \mu_P(a+c, a+d) &\geq \mu_P(c,d) \\ \nu_P(a+c, a+d) &\leq \nu_P(c,d) \\ \mu_P(xa, xb) &\geq \mu_P(a,b) \\ \nu_P(xa, xb) &\leq \nu_P(a,b) \end{aligned}$$

(ii) P is called an intuitionistic L-fuzzy right congruence on R, (ILFRC (R)) if, for all a, b, c, d, $x \in R$,

$$\begin{aligned} \mu_P(a+c, b+c) &\geq \mu_P(a,b) \\ \nu_P(a+c, b+c) &\leq \nu_P(a,b) \\ \mu_P(ax, bx) &\geq \mu_P(a,b) \\ \nu_P(ax, bx) &\leq \nu_P(a,b) \end{aligned}$$

(iii) P is called an intuitionistic L-fuzzy congruence on R, (ILFC (R)) if, for all $a, b, c, d, x, y \in R$

$$\begin{aligned} \mu_P(a+c, b+d) &\geq \mu_P(a,b) \wedge \mu_P(c,d) \\ \nu_P(a+c, b+d) &\leq \nu_P(a,b) \vee \nu_P(c,d) \\ \mu_P(ax, by) &\geq \mu_P(a,b) \wedge \mu_P(x,y) \\ \nu_P(ax, by) &\leq \nu_P(a,b) \vee \nu_P(x,y) \end{aligned}$$

Clearly $\Delta, \nabla \in$ ILFC (R).

The following proposition is immediate.

Proposition 14. *Let $P = \{\langle(x,y), \mu_P(x,y), \nu_P(x,y)\rangle/(x,y) \in R \times R\}$ be an ILFER (R). Then P is an ILFC (R), if and only if P is ILFLC (R) and ILFRC (R).*

Proposition 15. *Let*

$$P = \{\langle(x,y), \mu_P(x,y), \nu_P(x,y)\rangle/(x,y) \in R \times R\}$$

and

$$Q = \{\langle(x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle/(x,y) \in R \times R\}$$

be ILFC (R). If $P \circ Q = Q \circ P$ then $P \circ Q$ is an ILFC (R).

Proof. Clearly $P \circ Q \in$ ILFER (R). Let $a, b, c, d \in R$. Then for each $z, y \in R$

$$\begin{aligned} \mu_{P\circ Q}(a+c, a+d) &= \vee_{z\in R}(\mu_P(a+c, z) \wedge \mu_Q(z, a+d)) \\ &\geq \mu_P(a+c, z+y) \wedge \mu_Q(z+y, a+d) \\ &\geq [\mu_P(a,z) \wedge \mu_P(c,y)] \wedge [\mu_Q(z,a) \wedge \mu_Q(y,d)] \\ &= [\mu_P(a,z) \wedge \mu_Q(z,a)] \wedge [\mu_P(c,y) \wedge \mu_Q(y,d)]. \end{aligned}$$

Hence

$$\begin{aligned} \mu_{P\circ Q}(a+c, a+d) &\geq (\vee_{z\in R}[\mu_P(a,z) \wedge \mu_Q(z,a)]) \\ &\qquad \wedge (\vee_{y\in R}[\mu_P(c,y) \wedge \mu_Q(y,d)]) \\ &= \mu_{P\circ Q}(a,a) \wedge \mu_{P\circ Q}(c,d) \\ &= \mu_{P\circ Q}(c,d). \end{aligned}$$

Let $a, b, c, d \in R$. Then for each $y, z \in R$

$$\begin{aligned} \nu_{P\circ Q}(a+c, a+d) &= \wedge_{z\in R}(\nu_P(a+c, z) \vee \nu_Q(z, a+d)) \\ &\leq \nu_P(a+c, z+y) \vee \nu_Q(z+y, a+d) \\ &\leq [\nu_P(a,z) \vee \nu_P(c,y)] \vee [\nu_Q(z,a) \vee \nu_Q(y,d)] \\ &= [\nu_P(a,z) \vee \nu_Q(z,a)] \vee [\nu_P(c,y) \vee \nu_Q(y,d)]. \end{aligned}$$

Hence

$$\begin{aligned} \nu_{P\circ Q}(a+c, a+d) &\leq (\wedge_{z\in R}[\nu_P(a,z) \vee \nu_Q(z,a)]) \\ &\qquad \vee (\wedge_{y\in R}[\nu_P(c,y) \vee \nu_Q(y,d)]) \\ &= \nu_{P\circ Q}(a,a) \vee \nu_{P\circ Q}(c,d) \\ &= \nu_{P\circ Q}(c,d). \end{aligned}$$

Let $a, x, y \in R$. Then

$$\begin{aligned} \mu_{P\circ Q}(ax, ay) &= \vee_{z\in R}[\mu_P(ax, z) \wedge \mu_Q(z, ay)] \\ &\geq \mu_P(ax, at) \wedge \mu_Q(at, ay) \\ &\geq \mu_p(x,t) \wedge \mu_Q(t,y) \end{aligned}$$

for each $t \in R$.

Hence

$$\begin{aligned} \mu_{P\circ Q}(ax, ay) &\geq \vee_{t\in R}(\mu_P(x,t) \wedge \mu_Q(t,y)) \\ &= \mu_{P\circ Q}(x,y). \end{aligned}$$

Let $a, x, y \in R$. Then

$$\begin{aligned} \nu_{P\circ Q}(ax, ay) &= \wedge_{z\in R}[\nu_P(ax, z) \vee \nu_Q(z, ay)] \\ &\leq \nu_P(ax, at) \vee \nu_Q(at, ay) \\ &\leq \nu_p(x,t) \vee \nu_Q(t,y) \end{aligned}$$

for each $t \in R$. Hence

$$\begin{aligned}\nu_{P\circ Q}(ax, ay) &\leq \wedge_{t\in R}[\nu_P(x,t) \vee \nu_Q(t,y)] \\ &= \nu_{P\circ Q}(x,y).\end{aligned}$$

Hence $P \circ Q$ is ILFLC (R). Similarly, $P \circ Q$ is ILFRC (R) and therefore $P \circ Q$ is an ILFC (R). □

The set of all ILFC (R) is a partially ordered set by the inclusion relation "$\subseteq$". For $P, Q \in$ ILFC (R), $P \cap Q$ is the greatest lower bound of P and Q and $P \cap Q \in$ ILFC (R). But $P \cup Q$ need not be an ILFC (R).

Proposition 16. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFC (R)*. Then* $P \cap Q \in$ *ILFC* (R)*.*

Proof. Clearly $P \cap Q \in$ ILFER (R). Let $a, b, c, d \in R$. Then

$$\begin{aligned}\mu_{P\wedge Q}(a+b, c+d) &= \mu_P(a+b, c+d) \wedge \mu_Q(a+b, c+d) \\ &\geq \mu_P(a,c) \wedge \mu_P(b,d) \wedge \mu_Q(a,c) \wedge \mu_Q(b,d) \\ &= \mu_{P\wedge Q}(a,c) \wedge \mu_{P\wedge Q}(b,d)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\vee Q}(a+b, c+d) &= \nu_P(a+b, c+d) \vee \nu_Q(a+b, c+d) \\ &\leq \nu_P(a,c) \vee \nu_P(b,d) \vee \nu_Q(a,c) \vee \nu_Q(b,d) \\ &= \nu_{P\vee Q}(a,c) \vee \nu_{P\vee Q}(b,d).\end{aligned}$$

For $x, y, a, b \in R$

$$\begin{aligned}\mu_{P\wedge Q}(ax, by) &= \mu_P(ax, by) \wedge \mu_Q(ax, by) \\ &\geq \mu_P(a,b) \wedge \mu_P(x,y) \wedge \mu_Q(a,b) \wedge \mu_Q(x,y) \\ &= \mu_{P\wedge Q}(a,b) \wedge \mu_{P\wedge Q}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\vee Q}(ax, by) &= \nu_P(ax, by) \vee \nu_Q(ax, by) \\ &\leq \nu_P(a,b) \vee \nu_P(x,y) \vee \nu_Q(a,b) \vee \nu_Q(x,y) \\ &= \nu_{P\vee Q}(a,b) \vee \nu_{P\vee Q}(x,y).\end{aligned}$$

Hence $P \cap Q \in$ ILFC (R). □

5 Lattice of intuitionistic L-fuzzy congruence

In this section, the lattice structure of intuitionistic L-fuzzy congruence on a ring is studied. We prove that the intuitionistic L-fuzzy congruence on a ring forms a complete modular lattice.

The following lemma is straightforward.

Lemma 1. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFC (R). *Define* $P \vee Q = (P \cup Q)^\infty = \cup_{n\in N}(P \cup Q)^n$. *Then,* $P \vee Q \in$ *ILFC* (R).

On the set of ILFC (R), we define the following binary operations $\vee$ and $\wedge$ as follows.

For $P, Q \in$ ILFC (R),

$$P \vee Q = (P \cup Q)^e = (P \circ Q)^\infty \text{ and } P \wedge Q = P \cap Q.$$

Then the following result is immediate.

Theorem 4. *The set* $($*ILFC*$(R), \wedge, \vee)$ *is a complete lattice with* Δ *as the least element and* ∇ *as the greatest element.*

Proposition 17. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be ILFC (R). *Then,* $P \circ Q = Q \circ P$.

Proof. Let $x, y \in R$. Then

$$\begin{aligned}
\mu_{P\circ Q}(x,y) &= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y)] \\
&= \vee_{z\in R}[(\mu_P(y,y) \wedge \mu_P(-z,-z) \wedge \mu_P(x,z)) \\
&\qquad \wedge (\mu_Q(z,y) \wedge \mu_Q(-z,-z) \wedge \mu_Q(x,x))] \\
&\leq \vee_{z\in R}[\mu_P(y-z+x, y-z+z) \wedge (\mu_Q(z-z+x, y-z+x))] \\
&= \vee_{z\in R}[\mu_Q(x, y-z+x) \wedge \mu_P(y-z+x, y)] \\
&= \mu_{Q\circ P}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{P\circ Q}(x,y) &= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y)] \\
&= \wedge_{z\in R}[(\nu_P(y,y) \vee \nu_P(-z,-z) \vee \nu_P(x,z)) \\
&\qquad \vee (\nu_Q(z,y) \vee \nu_Q(-z,-z) \vee \nu_Q(x,x))] \\
&\geq \wedge_{z\in R}[\nu_P(y-z+x, y-z+z) \vee (\nu_Q(z-z+x, y-z+x))] \\
&= \wedge_{z\in R}[\nu_Q(x, y-z+x) \vee \nu_P(y-z+x, y)] \\
&= \nu_{Q\circ P}(x,y).
\end{aligned}$$

Hence $P \circ Q \subseteq Q \circ P$. Also $Q \circ P \subseteq P \circ Q$. Hence $P \circ Q = Q \circ P$. $\square$

Theorem 5. *Let*

$$P = \{\langle (x,y), \mu_P(x,y), \nu_P(x,y)\rangle / (x,y) \in R \times R\}$$

and

$$Q = \{\langle (x,y), \mu_Q(x,y), \nu_Q(x,y)\rangle / (x,y) \in R \times R\}$$

be an ILFC (R)*, then* $P \circ Q = P \vee Q$*.*

The proof is direct from previous results.

Definition 12. A lattice $(L, \leq, \wedge, \vee)$ is said to be modular if for any $x, y, z \in L$ and $x \leq z$, then

$$(x \vee y) \wedge z = x \vee (y \wedge z).$$

For $x, y, z \in L$ if $x \leq z$ then

$$x \vee (y \wedge z) \leq (x \vee y) \wedge z.$$

This inequality is called the modular inequality.

Theorem 6. *The lattice ILFC* (R) *is modular.*

Proof. Let $P, Q, N \in$ ILFC (R) such that $P \subseteq N$. Then by modular inequality

$$P \vee (Q \wedge N) \subseteq (P \vee Q) \wedge N$$

holds. Then for $x, y \in R$,

$$\begin{aligned}
\mu_{P\vee(Q\wedge N)}(x,y) &= \mu_{P\circ(Q\cap N)}(x,y) \\
&= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_{Q\wedge N}(z,y)] \\
&= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y) \wedge \mu_N(z,y)] \\
&= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y) \wedge \mu_{N\circ N}(z,y)] \\
&\geq \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y) \wedge \mu_N(z,x) \wedge \mu_N(x,y)] \\
&= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y) \wedge \mu_N(x,z) \wedge \mu_N(x,y)] \\
&\geq \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y) \wedge \mu_P(x,z) \wedge \mu_N(x,y)] \\
&= \vee_{z\in R}[\mu_P(x,z) \wedge \mu_Q(z,y)] \wedge \mu_N(x,y) \\
&= \mu_{(P\circ Q)\cap N}(x,y) \\
&= \mu_{(P\vee Q)\wedge N}(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{P\vee(Q\wedge N)}(x,y) &= \nu_{P\circ(Q\cap N)}(x,y) \\
&= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_{Q\vee N}(z,y)] \\
&= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y) \vee \nu_N(z,y)] \\
&= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y) \vee \nu_{N\circ N}(z,y)] \\
&\leq \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y) \vee \nu_N(z,x) \vee \nu_N(x,y)] \\
&= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y) \vee \nu_N(x,z) \vee \nu_N(x,y)] \\
&\leq \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y) \vee \nu_P(x,z) \vee \nu_N(x,y)]
\end{aligned}$$

$$
\begin{aligned}
&= \wedge_{z\in R}[\nu_P(x,z) \vee \nu_Q(z,y)] \vee \nu_N(x,y) \\
&= \nu_{(P\circ Q)\cap N}(x,y) \\
&= \nu_{(P\vee Q)\wedge N}(x,y).
\end{aligned}
$$

Hence $P \vee (Q \wedge N) \supseteq (P \vee Q) \wedge N$. Hence ILFC (R) is a modular lattice. □

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