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INTUITIONISTIC FUZZY IDEALS IN SEMIRINGS

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Abstract. The purpose of this paper is to introduce the notions of intuitionistic fuzzy ideals of a semiring and equivalence relations on the family of all intuitionistic fuzzy ideals of a semiring and investigate some related properties.

1. Introduction

Following the introduction of fuzzy sets by L. A. Zadeh ([17]), the fuzzy set theory developed by Zadeh himself and others can be found in mathematics and many applied areas. In 1982, W. Liu ([10]) defined and studied fuzzy subrings as well as fuzzy ideals in rings. Subsequently, T. K. Mukherjee and M. K. Sen ([12]), K. L. N. Swamy and U. M. Swamy ([14]), and Zhang Yue ([16]) fuzzified certain standard concepts/results on rings and ideals. The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1,2,5,10]). The present two authors with J. Neggers ([7]) extended the concept of an L-fuzzy (characteristic) left (resp. right) ideal of a ring to a semiring R, and showed that each level left (resp. right) ideal of L-fuzzy left (resp. right) ideal μ of R is characteristic iff μ is L-fuzzy characteristic. Moreover, they discussed the notion of normal L-fuzzy ideals in semirings ([8]), and obtained some properties of L-fuzzy ideals related to level ideals in semirings ([13]). The intuitionistic fuzzy set was introduced by K. T. Atanassove ([3]), as a generalization of fuzzy sets. It was applied to other areas: near-rings ([15]), incline algebras ([6]). In this paper, we apply the concepts of intuitionistic fuzzy sets to ideals of semirings and introduce the notions of intuitionistic fuzzy ideals of a semiring and equivalence relations on the family of all intuitionistic fuzzy ideals of a semiring and investigate some related properties.

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2. Preliminaries

By a *semiring* ([2]) we shall mean a set R endowed with two associative binary operations called *addition* and *multiplication* (denoted by $+$ and $\cdot$, respectively) satisfying the following conditions:

(i) addition is a commutative operation,
(ii) there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$,
(iii) multiplication distributes over addition both from the left and from the right.

Now, we review the concepts of fuzzy sets and intuitionistic fuzzy sets (see [3, 4, 17]). Let X be a non-empty set. A map $\mu : X \to [0, 1]$ is called a *fuzzy set* in X, and the *complement* of a fuzzy set μ in X, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Let X and Y be two non-empty sets and $f : X \to Y$ be a function, and let μ and ν be any fuzzy sets in X and Y respectively. The *image of* μ *under* f, denoted by $f(\mu)$, is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for each $y \in Y$. The *preimage of* ν *under* f, denoted by $f^{-1}(\nu)$, is a fuzzy set in X defined by $(f^{-1}(\nu))(x) = \nu(f(x))$ for each $x \in X$.

An *intuitionistic fuzzy set* (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership, respectively, satisfying the following condition:

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$.

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in X can be identified with an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$. Clearly, every fuzzy set μ in X is an IFS of the form $(\mu, \overline{\mu})$.

Definition 2.1. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets in X. Then

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(3) $\overline{A} = (\gamma_A, \mu_A)$,
(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$,

(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$,
(6) $\Box A = (\mu_A, \overline{\mu_A})$,
(7) $\Diamond A = (\overline{\gamma_A}, \gamma_A)$.

Let X and Y be two non-empty sets and $f : X \to Y$ be a function. If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy set in Y, then the *preimage of B under f*, denoted by $f^{-1}(B)$, is an IFS in X defined by $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy set in X, then the *image of A under f*, denoted by $f(A)$, is an IFS in Y defined by

$$f(A) = (f(\mu_A), f_{-}(\gamma_A)),$$

where

$$f_{-}(\gamma_A)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

for each $y \in Y$.

Definition 2.2. A fuzzy set $\mu(\in \mathcal{F}(R))$ is called a *fuzzy left* (resp.*right*) *ideal* of R if for all $x, y, r \in R$,
(F1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
(F2) $\mu(rx) \geq \mu(x)$ (resp. $\mu(xr) \geq \mu(x)$).

A fuzzy set μ is a *fuzzy ideal* of R if and only if it is both fuzzy left and right ideal of R. It follows from the definition of the semiring that if μ is an L-fuzzy left (resp. right) ideal of R, then $\mu(0) \geq \mu(x)$ for all $x \in X$. As the idea of a semiring is a generalization of the idea of a ring, the notion of fuzzy left (resp. right) ideal of a semiring is also a generalization of the notion of L-fuzzy left (resp. right) ideal in rings. Hence, every fuzzy left (resp. right) ideal of a ring is a fuzzy left (resp. right) ideal of a semiring. But the converse need not at all be true. (See [7]).

3. Intuitionistic fuzzy ideals of semirings

Definition 3.1. An IFS $A = (\mu_A, \gamma_A)$ of R is called an *intuitionistic fuzzy subsemiring* of R if for all $x, y \in R$,

(IF1) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x + y) \leq \gamma_A(x) \vee \gamma_A(y)$,
(IF2) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$.

Definition 3.2. An IFS $A = (\mu_A, \gamma_A)$ of R is called an *intuitionistic fuzzy ideal* of R if A satisfies (IF1) and for all $x, y, r \in R$,

(IFR1) $\mu_A(rx) \geq \mu_A(x)$ and $\gamma_A(rx) \leq \gamma_A(x)$,
(IFR2) $\mu_A(xr) \geq \mu_A(x)$ and $\gamma_A(xr) \leq \gamma_A(x)$.

If $A = (\mu_A, \gamma_A)$ satifies (IF1) and (IFR1), then A is called an *intuitionistic fuzzy left ideal* of R, and if $A = (\mu_A, \gamma_A)$ satifies (IF1) and (IFR2), then A is called an *intuitionistic fuzzy right ideal* of R.

Example 3.3. Let $R := \{a, b, c, d\}$ be a set with two binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	c

·	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	b	b
d	a	b	b	b

Then $(R, +, \cdot)$ is a semiring ([7]). We define an IFS $A = (\mu_A, \gamma_A)$ by

$$\mu_A(a) = 1, \mu_A(b) = \frac{2}{3}, \mu_A(c) = \frac{1}{3}, \mu_A(d) = 0,$$

$$\gamma_A(a) = 0, \gamma_A(b) = \frac{1}{3}, \gamma_A(c) = \frac{1}{3}, \gamma_A(d) = 1.$$

Then A is an intuitionistic fuzzy ideal of R.

Lemma 3.4. *If an IFS $A = (\mu_A, \gamma_A)$ in R satisfies (IF1), then $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in R$.* □

Lemma 3.5. *Every intuitionistic fuzzy ideal in R is an intuitionistic fuzzy subsemiring of R.*

Proof. It follows immediately from the Definitions 3.1 and 3.2. □

Theorem 3.6. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are intuitionistic fuzzy ideals (subsemirings) of R, then so is $A \cap B$.*

Proof. For any $x, y \in R$, we have that

$$\begin{aligned}(\mu_A \wedge \mu_B)(x + y) &= \mu_A(x + y) \wedge \mu_B(x + y) \\ &\geq (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)) \\ &= (\mu_A \wedge \mu_B)(x) \wedge (\mu_A \wedge \mu_B)(y),\end{aligned}$$

$$\begin{aligned}(\gamma_A \vee \gamma_B)(x + y) &= \gamma_A(x + y) \vee \gamma_B(x + y) \\ &\leq (\gamma_A(x) \vee \gamma_B(x)) \vee (\gamma_A(y) \vee \gamma_B(y)) \\ &= (\gamma_A \vee \gamma_B)(x) \vee (\gamma_A \vee \gamma_B)(y),\end{aligned}$$

and if $x, r \in R$, then we have that

$$(\mu_A \wedge \mu_B)(xr) = \mu_A(xr) \wedge \mu_B(xr) \geq \mu_A(x) \wedge \mu_B(x) = (\mu_A \wedge \mu_B)(x),$$

$$(\gamma_A \vee \gamma_B)(xr) = \gamma_A(xr) \vee \gamma_B(xr) \leq \gamma_A(x) \vee \gamma_B(x) = (\gamma_A \vee \gamma_B)(x).$$

Similarly, we get $(\mu_A \wedge \mu_B)(rx) \geq (\mu_A \wedge \mu_B)(x)$ and $(\gamma_A \vee \gamma_B)(rx) \leq (\gamma_A \vee \gamma_B)(x)$ for all $x, r \in R$. Hence $A \cap B$ is an intuitionistic fuzzy ideal of R. We can prove for intuitionistic fuzzy subsemirings, and omit the proof. □

Lemma 3.7. *Let* $A = (\mu_A, \gamma_A)$ *be an IFS in* R. *Then* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal (resp. subsemiring) if and only if* μ_A *and* $\overline{\gamma_A}$ *are fuzzy ideals (resp. subsemirings) of* R.

Proof. It follows from the definitions. $\square$

Theorem 3.8. *Let* $A = (\mu_A, \gamma_A)$ *be an IFS in* R. *Then* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal (resp. subsemiring) in* R *if and only if* $\Box A = (\mu_A, \overline{\mu_A})$ *and* $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ *are intuitionistic fuzzy ideals (resp. subsemirings) in* R.

Proof. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal in R, then $\mu_A = \overline{\overline{\mu_A}}$ and $\overline{\gamma_A}$ are fuzzy ideals of R from Lemma 3.7, hence $\Box A = (\mu_A, \overline{\mu_A})$ and $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ are intuitionistic fuzzy ideals of R from Lemma 3.7. Conversely, if $\Box A = (\mu_A, \overline{\mu_A})$ and $\Diamond A = (\overline{\gamma_A}, \gamma_A)$ are intuitionistic fuzzy ideals in R, then the fuzzy sets μ_A and $\overline{\gamma_A}$ are fuzzy ideals in R, hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal in R. $\square$

Theorem 3.10. *Let* R *and* S *be two semirings and* $f : R \to S$ *an onto homomorphism. If an IFS* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal (resp. subsemiring) in* R, *then the image* $f(A) = (f(\mu_A), f_-(\gamma_A))$ *of* A *under* f *is an intuitionistic fuzzy ideal (resp. subsemiring) in* S.

Proof. If $f : R \to S$ is an onto homomorphism and $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal (resp. subsemiring) in R, then $\{x \in R \mid x \in f^{-1}(y_1 + y_2)\} \supseteq \{x_1 + x_2 \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$ and $\{z \in R \mid z \in f^{-1}(sy)\} \supseteq \{rx \mid r \in f^{-1}(s), x \in f^{-1}(y)\}$, for any $y_1, y_2, s, y \in S$, hence $f(\mu_A)(y_1 + y_2) \geq f(\mu_A)(y_1) \wedge f(\mu_A)(y_2)$, $f_-(\gamma_A)(y_1 + y_2) \leq f_-(\gamma_A)(y_1) \vee f_-(\gamma_A)(y_2)$ and $f(\mu_A)(sy) \geq f(\mu_A)(y)$, $f_-(\gamma_A)(sy) \leq f_-(\gamma_A)(y)$ for all $y_1, y_2, s, y \in S$. We can prove (IFR2) similarly. Hence $f(A) = (f(\mu_A), f_-(\gamma_A))$ is an intuitionistic fuzzy ideal in S. $\square$

Theorem 3.10. *Let* R *and* S *be two semirings and* $f : R \to S$ *a homomorphism. If an IFS* $B = (\mu_B, \gamma_B)$ *is an intuitionistic fuzzy ideal (resp. subsemiring) in* S, *then the preimage* $f^{-1}(A) = (f^{-1}(\mu_B), f^{-1}(\gamma_A))$ *of* B *under* f *is an intuitionistic fuzzy ideal (resp. subsemiring) in* R.

Proof. It follows immediately from the definitions. $\square$

4. An equivalence relation on intuitionistic fuzzy subsemiring

For any $\alpha \in [0, 1]$ and fuzzy set μ in a non-empty set X, the set $U(\mu; \alpha) = \{x \in R \mid \mu(x) \geq \alpha\}$ is called an *upper* α*-level cut* of μ and the set $L(\mu; \alpha) = \{x \in R \mid \mu(x) \leq \alpha\}$ is called a *lower* α*-level cut* of μ.

Theorem 4.1. *Let $A = (\mu_A, \gamma_A)$ be an IFS in R. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal (resp. subsemiring) if and only if $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are ideals (resp. subsemirings) of R for any $\alpha \in [0, \mu_A(0)]$ and $\beta \in [\gamma_A(0), 1]$.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal in R and $\alpha \in [0, \mu_A(0)]$. If $x, y \in U(\mu_A; \alpha)$, then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$, hence $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$. It follows that $x + y \in U(\mu_A; \alpha)$. If $r \in R$ and $x \in U(\mu_A; \alpha)$, then $\mu_A(x) \geq \alpha$, whence we have that $\mu_A(rx) \geq \mu_A(x) \geq \alpha$ and $\mu_A(xr) \geq \mu_A(x) \geq \alpha$, hence $rx, xr \in U(\mu_A; \alpha)$. That is, $RU(\mu_A; \alpha) \subseteq U(\mu_A; \alpha)$ and $U(\mu_A; \alpha)R \subseteq U(\mu_A; \alpha)$. We can prove similarly for $L(\gamma_A; \beta)$. Conversely, let $x, y \in R$ and let $\alpha = \mu_A(x) \wedge \mu_A(y)$. Then $x, y \in U(\mu_A; \alpha)$, and $x + y \in U(\mu_A; \alpha)$, since $U(\mu_A; \alpha)$ is an ideal of R. Hence $\mu_A(x + y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$. If $x, r \in R$ and $\alpha = \mu_A(x)$, then $x \in U(\mu_A; \alpha)$, and $rx, xr \in U(\mu_A; \alpha)$, since $U(\mu_A; \alpha)$ is an ideal in R. Hence $\mu_A(rx) \geq \alpha = \mu_A(x)$ and $\mu_A(xr) \geq \alpha = \mu_A(x)$. We can prove similarly for subsemiring, and we omit the proof. □

If H is a subsemiring (resp. ideal) of R, then the IFS $\mathcal{H} = (\chi_H, \overline{\chi_H})$ is an intuitionistic fuzzy subsemiring (resp. ideal) of R from Theorem 4.1, where χ_H is the characteristic function of H as follows:

$$\chi_H(x) = \begin{cases} 1 & \text{if } x \in H, \\ 0 & \text{if otherwise}, \end{cases}$$

for each $x \in R$.

Let $IFSN(R)$ be the family of all intuitionistic fuzzy subsemirings of R and let α be a fixed real number in $[0, 1]$. We define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFSN(R)$ as follows:

$$(A, B) \in \mathfrak{U}^\alpha \Leftrightarrow U(\mu_A; \alpha) = U(\mu_B; \alpha),$$

and

$$(A, B) \in \mathfrak{L}^\alpha \Leftrightarrow L(\gamma_A; \alpha) = L(\gamma_B; \alpha),$$

respectively, for any $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in IFSN(R)$. Then the two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations on $IFSN(R)$. These equivalence relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFSN(R)$ give rise to partitions of $IFSN(R)$ into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A, \gamma_A) \in IFSN(R)$, respectively. And we will denote the quotient sets of $IFSN(R)$ by $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ as $IFSN(R)/\mathfrak{U}^\alpha$ and $IFSG(R)/\mathfrak{L}^\alpha$, respectively.

If $SN(R)$ is the family of all subsemirings of R and $\alpha \in [0, 1]$, then we define two maps U_α and L_α from $IFSN(R)$ to $SN(R) \cup \{\emptyset\}$ as follows:

$$U_\alpha(A) = U_\alpha(\mu_A, \gamma_A) = U(\mu_A; \alpha),$$

and

$$L_\alpha(A) = L_\alpha(\mu_A, \gamma_A) = L(\gamma_A; \alpha),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFSN(R)$. Then the maps U_α and L_α are well-defined.

Theorem 4.2. *For any $\alpha \in (0,1)$, the maps U_α and L_α are surjective from $IFSN(R)$ onto $SN(R) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. If $0_\sim = (0,1)$, then $0_\sim$ is an intuitionistic fuzzy subsemiring in R, from Theorem 4.1, and $U_\alpha(0_\sim) = L_\alpha(0_\sim) = \emptyset$. If H is a subsemiring of R, then for the intuitionistic fuzzy subsemiring $\mathcal{H} = (\chi_H, \overline{\chi_H})$, $U_\alpha(\mathcal{H}) = U(\chi_H;\alpha) = H$ and $L_\alpha(\mathcal{H}) = L(\overline{\chi_H};\alpha) = H$. Hence U_α and L_α are surjective. □

Let $IFSG(R)$ be the family of all intuitionistic fuzzy ideals of R and $SG(R)$ the family of all ideals of R. Then $IFSG(R) \subseteq IFSN(R)$ from Lemma 3.6 and $SG(R) \subseteq SN(R)$.

Corollary 4.3. *If the maps U_α^* and L_α^* are the restrictions of U_α and L_α to $IFSG(R)$, where $\alpha \in (0,1)$, then U_α^* and L_α^* are surjective from $IFSG(R)$ onto $SG(R) \cup \{\emptyset\}$.*

Proof. If $H \in SG(R)$, then for any $\mathcal{H} = (\chi_H, \overline{\chi_H}) \in IFSG(R)$, $U_\alpha(\mathcal{H}) = L_\alpha(\mathcal{H}) = H$, whence $H \in Im(U_\alpha)$ and $H \in Im(L_\alpha)$, and $U_\alpha(0_\sim) = L_\alpha(0_\sim) = \emptyset$ for $0_\sim = (0,1) \in IFSG(R)$. Hence $SG(R) \cup \{\emptyset\} \subseteq Im(U_\alpha)$ and $SG(R) \cup \{\emptyset\} \subseteq Im(L_\alpha)$. And $Im(U_\alpha) \subseteq SG(R) \cup \{\emptyset\}$ and $Im(L_\alpha) \subseteq SG(R) \cup \{\emptyset\}$ from Theorem 4.1 and the fact that $\emptyset$ is in $Im(U_\alpha)$ and $Im(L_\alpha)$. □

Theorem 4.4. *The quotient sets $IFSN(R)/\mathfrak{U}^\alpha$ and $IFSN(R)/\mathfrak{L}^\alpha$ are equipotent to $SN(R) \cup \{\emptyset\}$ for any $\alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{U_\alpha} : IFSN(R)/\mathfrak{U}^\alpha \to SN(R) \cup \{\emptyset\}$ and $\overline{L_\alpha} : IFSN(R)/\mathfrak{L}^\alpha \to SN(R) \cup \{\emptyset\}$ be the maps defined by

$$\overline{U_\alpha}([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A) \text{ and } \overline{L_\alpha}([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFSN(R)$. If $U(\mu_A;\alpha) = U(\mu_B;\alpha)$ and $L(\gamma_A;\alpha) = L(\gamma_B;\alpha)$ for $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in IFSN(R)$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$. Hence the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are injective. To show that the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective, let H be a subsemiring of R. Then for $\mathcal{H} = (\chi_H, \overline{\chi_H}) \in IFSN(R)$, $\overline{U_\alpha}([\mathcal{H}]_{\mathfrak{U}^\alpha}) = U(\chi_H;\alpha) = H$ and $\overline{L_\alpha}([\mathcal{H}]_{\mathfrak{L}^\alpha}) = L(\overline{\chi_H};\alpha) = H$. And for $0_\sim = (0,1) \in IFSN(R)$, $\overline{U_\alpha}([0_\sim]_{\mathfrak{U}^\alpha}) = U(0;\alpha) = \emptyset$ and $\overline{L_\alpha}([0_\sim]_{\mathfrak{L}^\alpha}) = L(1;\alpha) = \emptyset$. Hence the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective. □

Corollary 4.5. *If $\mathfrak{U}^\alpha_{IFSG(R)}$ and $\mathfrak{L}^\alpha_{IFSG(R)}$ are the restrictions of the equivalence relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, respectively, to $IFSG(R)$ where $\alpha \in (0,1)$, then the quotient sets $IFSG(R)/\mathfrak{U}^\alpha_{IFSG(R)}$ and $IFSG(R)/\mathfrak{L}^\alpha_{IFSG(R)}$ are equipotent to $SG(R) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. If $\overline{U_\alpha^*} : IFSG(R)/\mathfrak{U}^\alpha_{IFSG(R)} \to SG(R) \cup \{\emptyset\}$ and $\overline{L_\alpha^*} : IFSG(R)/\mathfrak{L}^\alpha_{IFSG(R)} \to SG(R) \cup \{\emptyset\}$ are the maps defined by

$$\overline{U_\alpha^*}([A]_{\mathfrak{U}^\alpha_{IFSG(R)}}) = U_\alpha^*(A) \text{ and } \overline{L_\alpha^*}([A]_{\mathfrak{L}^\alpha_{IFSG(R)}}) = L_\alpha^*(A),$$

respectively, for each $A = (\mu_A, \gamma_A) \in IFSN(R)$, then $\overline{U_\alpha^*}$ and $\overline{L_\alpha^*}$ are bijective maps. The proof is similar to the proof of Theorem 4.4, and omit the proof. □

For any $\alpha \in [0,1]$, we define another relation $\mathfrak{R}^\alpha$ on $IFSN(R)$ as follows:

$$(A,B) \in \mathfrak{R}^\alpha \Leftrightarrow U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha)$$

for any $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in IFSN(R)$. Then the relation $\mathfrak{R}^\alpha$ is also an equivalence relation on $IFSN(R)$.

Theorem 4.6. *For any $\alpha \in (0,1)$, if $I_\alpha : IFSN(R) \to SN(R) \cup \{\emptyset\}$ is a map defined by*

$$I_\alpha(A) = U_\alpha(A) \cap L_\alpha(A)$$

for each $A = (\mu_A, \gamma_A) \in IFSN(R)$, then the map I_α is surjective.

Proof. Let $\alpha \in (0,1)$. For $0_\sim = (0,1) \in IFSN(R)$, $I_\alpha(0_\sim) = U_\alpha(0_\sim) \cap L_\alpha(0_\sim) = U(0;\alpha) \cap L(1;\alpha) = \emptyset$. And for any $H \in SN(R)$, there exists $\mathcal{H} = (\chi_H, \overline{\chi_H}) \in IFSN(R)$ such that $I_\alpha(\mathcal{H}) = U(\chi_H;\alpha) \cap L(\overline{\chi_H};\alpha) = H$. □

Corollary 4.7. *If I_α^* is the restriction of I_α to $IFSG(R)$, then I_α^* is surjective map from $IFSG(R)$ to $SG(R) \cup \{\emptyset\}$.*

Proof. If $H \in SG(R)$, then for any $\mathcal{H} = (\chi_H, \overline{\chi_H}) \in IFSG(R)$, $I_\alpha^*(\mathcal{H}) = I_\alpha(\mathcal{H}) = U_\alpha(\mathcal{H}) \cap L_\alpha(\mathcal{H}) = H$, hence $H \in Im(I_\alpha^*)$, and since $I_\alpha(0_\sim) = U_\alpha(0_\sim) \cap L_\alpha(0_\sim) = \emptyset$ for $0_\sim = (0,1) \in IFSG(R)$, $\emptyset \in Im(I_\alpha^*)$. It follows that $SG(R) \cup \{\emptyset\} \subseteq Im(I_\alpha^*)$. And $Im(I_\alpha^*) \subseteq SG(R) \cup \{\emptyset\}$ from Theorem 4.1. □

Theorem 4.8. *For any $\alpha \in (0,1)$, the quotient set $IFSN(R)/\mathfrak{R}^\alpha$ is equipotent to $SN(R) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$ and let $\overline{I_\alpha} : IFSN(R)/\mathfrak{R}^\alpha \to SN(R) \cup \{\emptyset\}$ be a map defined by

$$\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A)$$

for each $[A]_{\mathfrak{R}^\alpha} \in IFSG(R)/\mathfrak{R}^\alpha$. If $\overline{I_\alpha}([A]_{\mathfrak{R}^\alpha}) = \overline{I_\alpha}([B]_{\mathfrak{R}^\alpha})$ for any $[A]_{\mathfrak{R}^\alpha}, [B]_{\mathfrak{R}^\alpha} \in IFSG(R)/\mathfrak{R}^\alpha$, then $U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha)$, hence $(A,B) \in \mathfrak{R}^\alpha$, and $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. It follows that $\overline{I_\alpha}$ is injective. For $0_\sim = (0,1) \in IFSN(R)$, $\overline{I_\alpha}(0_\sim) = I_\alpha(0_\sim) = \emptyset$. If $H \in SN(R)$, then for $\mathcal{H} = (\chi_H, \overline{\chi_H}) \in IFSN(R)$, $\overline{I_\alpha}(\mathcal{H}) = I_\alpha(H) = H$. Hence $\overline{I_\alpha}$ is a bijective map. □

Corollary 4.9. *If $\alpha \in (0,1)$ and $\mathfrak{R}^{\alpha}_{IFSG(R)}$ is the restriction of the equivalence relation $\mathfrak{R}^{\alpha}$ to $IFSG(R)$, then $IFSG(R)/\mathfrak{R}^{\alpha}_{IFSG(R)}$ is equipotent to $SG(R) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. If $\overline{I^*_\alpha} : IFSG(R)/\mathfrak{R}^{\alpha}_{IFSG(R)} \to SG(R) \cup \{\emptyset\}$ is the map defined by

$$\overline{I^*_\alpha}([A]_{\mathfrak{U}^{\alpha}_{IFSG(R)}}) = I^*_\alpha(A)$$

for each $A = (\mu_A, \gamma_A) \in IFSN(R)$, then $\overline{I^*_\alpha}$ is bijective map from the similar way to the proof of Theorem 4.8. □

References

1. J. Ahsan and M. Shabir, *Semirings with projective ideals*, Math. Japonica **38** (1993), 271 - 276.
2. P. J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21** (1969), 412 - 416.
3. K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Sys. **20** (1986), 87-96.
4. K. T. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Sys. **61** (1994), 137-142.
5. L. Dale, *Direct sums of semirings and the Krull-Schmidt theorem*, Kyungpook Math. J. **17**, 135 - 141.
6. Y. B. Jun and H. S. Kim, *Intuitionistic fuzzy ideals of inclines*, Notes on Intuitionistic Fuzzy Sets **5 (2)** (1999), 33-41.
7. Y. B. Jun, J. Neggers and H. S. Kim, *On L-fuzzy ideals in semirings I*, Czech. Math. J. **48** (1998), 669-675.
8. Y. B. Jun, J. Neggers and H. S. Kim, *Normal L-fuzzy ideals in semirings*, Fuzzy Sets & Systems **82** (1996), 383-386.
9. Y. B. Jun and H. S. Kim, *Intuitionistic fuzzy ideals of inclines*, (submitted).
10. H. S. Kim, *On quotient semiring and extension of quotient halfring*, Comm. Korean Math. Soc. **4** (1989), 17-22.
11. Wang-jin Liu, *Fuzzy invariants subgroups and fuzzy ideals*, Fuzzy Sets and Sys. **8** (1987), 133 - 139.
12. T. K. Mukherjee and M. K. Sen, *On fuzzy ideals of a ring I*, Fuzzy Sets and Sys. **21** (1987), 99 - 104.
13. J. Neggers, Y. B. Jun and H. S. Kim, *On L-fuzzy ideals in semirings II*, Czech. Math. J. **49(124)** (1999), 127-133.
14. K. L. N. Swamy and U. M. Swamy, *Fuzzy prime idelas of rings*, J. Math. Anal. Appl. **134** (1988), 345 - 350.
15. Y. H. Yon, Y. B. Jun and K. H. Kim, *Intuitionistic fuzzy R-subgroups of near-ring*, Soochow J. Math. **27(3)** (2001), 243-253.
16. Zhang Yue, *Prime L-fuzzy ideals and primary L-fuzzy ideals*, Fuzzy Sets and Sys. **27** (1988), 345 - 350.
17. L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338 - 353.

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