NIFS 17 (2011), 1, 48-57

Intuitionistic fuzzy δ-connectedness and θ-connectedness

S. E. Abbas and Biljana Krsteska*

Department of Mathematics, Faculty of Science,
Sohag University, sohag 82524, Egypt.
*Faculty of Mathematics and Natural Science,
Univ. St. Cyril and Methodius Gazi
Baba b.b. P.O.162,1000 Skopje, Macedonia.

Dedicated to 90-th anniversary of Prof. Lotfi Zadeh

Abstract. We introduce and study the concepts of (r, s)-fuzzy δ-connected and (r, s)-fuzzy θ-connected for fuzzy sets in an intuitionistic fuzzy topological spaces in Šostak sense as a weaker version of (r, s)-fuzzy connected.
Keywords: Intuitionistic fuzzy topology, (r, s)-fuzzy δ-connected, (r, s)-fuzzy θ-connected
AMS Classification: 03E72, 54A40

1 Introduction and preliminaries

Kubiak [10] and Šostak [14] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [3], in the sense that not only the objects are fuzzified, but also the axiomatics. Chattopadhyay et al., [4] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [7-11].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [2]. By using intuitionistic fuzzy sets, Çoker and coworker [5,6] defined the topology of intuitionistic fuzzy sets. Recently, Samanta and Mondal [13], introduced the notion of intuitionistic gradation of openness of fuzzy sets, where to each fuzzy subsets there is a definite grade of openness and there is a grade of non-openness. Thus, the concept of gradation of openness and the topology of intuitionistic fuzzy sets.

In this paper, we present and investigate the notions of (r, s)-fuzzy δ-connectedness and (r, s)-fuzzy θ-connectedness relative to an intuitionistic fuzzy topological space in view of the definition of Šostak, and investigate the relationship with (r, s)-fuzzy connectedness. We compare all these forms of connectedness and investigate their properties in (r, s)-fuzzy almost regular, (r, s)-fuzzy semi-regular and (r, s)-fuzzy regular spaces.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A *fuzzy point* x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = t \text{if} y = x, \text{and} x_t(y) = 0 \text{if} y \neq x$. The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda \; q \; \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denote $\lambda \; \overline{q} \; \mu$.

Definition 1.1 [13] An intuitionistic gradation of openness (IGO, for short) on X is an ordered pair (τ, τ^*) of mappings from I^X to I such that

(IGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1, \forall \lambda \in I^X$,

(IGO2) $\tau(\overline{0}) = \tau(\underline{1}) = 1$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,

(IGO3) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$, for each $\lambda_i \in I^X, i = 1, 2$,

(IGO4) $\tau(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \tau^*(\lambda_i)$, for each $\lambda_i \in I^X, i \in \Delta$.

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (ifts, for short). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Theorem 1.2 ([1,12]) Let (X, τ, τ^*) be an ifts. For each $\lambda \in I^X, r \in I_0$ and $s \in I_1$, we define an operators $\mathcal{C} : I^X \times I_0 \times I_1 \to I^X, \mathcal{I} : I^X \times I_0 \times I_1 \to I^X$ as follows:

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\mu \mid \mu \geq \lambda, \tau(\underline{1} - \mu) \geq r, \tau^*(\underline{1} - \mu) \leq s\},$$

$$\mathcal{I}(\lambda, r, s) = \bigvee\{\mu \mid \mu \geq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq s\}.$$

2 (r, s)-fuzzy δ-cluster and (r, s)-fuzzy θ-cluster points

Definition 2.1 Let (X, τ, τ^*) be an ifts, $\mu \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then

(1) μ is called (r, s)-fuzzy $\mathcal{Q}$-neighborhood of x_t if $\tau(\mu) \geq r$, $\tau^*(\mu) \leq s$ and $x_t q \mu$.

(2) μ is called (r, s)-fuzzy $\mathcal{R}$-neighborhood of x_t if $\mu = \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)$ and $x_t q \mu$.

We denote

$\mathcal{Q}(x_t, r, s) = \{\mu \in I^X \mid \quad x_t q \mu, \quad \tau(\mu) \geq r, \quad \tau^*(\mu) \leq s\}$.

$\mathcal{R}(x_t, r, s) = \{\mu \in I^X \mid \quad x_t q \mu = \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)\}$.

Definition 2.2 Let (X, τ, τ^*) be an ifts, $\lambda \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then

(1) x_t is called (r, s)-fuzzy cluster point of λ if $\mu q \lambda$, for every $\mu \in \mathcal{Q}(x_t, r, s)$.

(2) x_t is called (r, s)-fuzzy δ-cluster point of λ if $\mu q \lambda$, for every $\mu \in \mathcal{R}(x_t, r, s)$.

(3) x_t is called (r, s)-fuzzy θ-cluster point of λ if $\mathcal{C}(\mu, r, s) q \lambda$, for every $\mu \in \mathcal{Q}(x_t, r, s)$.

We denote

$\mathrm{cl}(\lambda, r, s) = \bigvee\{x_t \in P_t(X) \mid \quad x_t \text{ is } (r, s) - \text{fuzzy cluster point of } \lambda\}$.

$\mathcal{D}(\lambda, r, s) = \bigvee\{x_t \in P_t(X) \mid \quad x_t \text{ is } (r, s) - \text{fuzzy } \delta - \text{ cluster point of } \lambda\}$.

$T(\lambda, r, s) = \bigvee\{x_t \in P_t(X) \mid \quad x_t \text{ is } (r, s) - \text{fuzzy } \theta - \text{ cluster point of } \lambda\}$.

Theorem 2.3 Let (X, τ, τ^*) be an ifts, $\lambda \in I^X$, $x_t \in P_t(X)$, $r \in I_0$ and $s \in I_1$. Then

(1) $\mathcal{C}(\lambda, r, s) = \mathrm{cl}(\lambda, r, s)$.

(2) $\mathcal{D}(\lambda, r, s) = \bigwedge\{\mu \mid \quad \lambda \leq \mu, \quad \mu = \mathcal{C}(\mathcal{I}(\mu, r, s), r, s)\}$.

(3) $T(\lambda, r, s) = \bigwedge\{\mu \mid \quad \lambda \leq \mathcal{I}(\mu, r, s), \quad \tau(\mu) \geq r, \tau^*(\mu) \leq s\}$.

(4) x_t is (r, s)-fuzzy δ-cluster (resp. (r, s)-fuzzy cluster and (r, s)-fuzzy θ-cluster) point of λ iff $x_t \in \mathcal{D}(\lambda, r, s)$ (resp. $x_t \in \mathcal{C}(\lambda, r, s)$ and $x_t \in T(\lambda, r, s)$).

(5) $\lambda \leq \mathcal{C}(\lambda, r, s) \leq \mathcal{D}(\lambda, r, s) \leq T(\lambda, r, s)$.

(6) If $\tau(\lambda) \geq r$ and $\tau^*(\lambda) \leq s$, then $\mathcal{C}(\lambda, r, s) = \mathcal{D}(\lambda, r, s) = \mathcal{T}(\lambda, r, s)$.

Proof. (1) and (3) are similarly proved as the following (2).

(2) Put $\rho = \bigwedge\{\mu \mid \quad \lambda \leq \mu, \quad \mu = \mathcal{C}(\mathcal{I}(\mu, r, s), r, s)\}$. Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{D}(\lambda, r, s) \not\geq \rho$. Then there exist $x \in X$ and $t \in (0,1)$ such that

$$\mathcal{D}(\lambda, r, s)(x) < t < \rho(x).$$

Since x_t is not (r,s)-fuzzy δ-cluster point of λ, there exists $\nu \in \mathcal{R}(x_t, r, s)$ such that $\lambda \leq \underline{1} - \nu$. Then $\lambda \leq \underline{1} - \nu = \mathcal{C}(\mathcal{I}(\underline{1} - \nu, r, s), r, s)$. Thus, $\rho \leq \underline{1} - \nu$. Furthermore, $x_t q \nu$ implies $\rho(x) \leq (\underline{1} - \nu)(x) < t$. It is a contradiction. Hence $\mathcal{D}(\lambda, r, s) \geq \rho$.

Suppose there exist $\lambda \in I^X$, $x \in X$ and $t \in (0,1)$ such that

$$\mathcal{D}(\lambda, r, s)(x) > t > \rho(x).$$

Then there exists $\mu \in I^X$ with $\lambda \leq \mu = \mathcal{C}(\mathcal{I}(\mu, r, s), r, s)$ such that

$$\mathcal{D}(\lambda, r, s)(x) > t > \mu(x) > \rho(x).$$

Then $\underline{1} - \mu \in \mathcal{R}(x_t, r, s)$ such that $\lambda \overline{q} (\underline{1} - \mu)$. Hence x_t is not (r,s)-fuzzy δ-cluster point of λ. It is a contradiction. Hence $\mathcal{D}(\lambda, r, s) \leq \rho$.

(4) ($\Rightarrow$) It is trivial.

($\Leftarrow$) Let x_t be not (r,s)-fuzzy δ-cluster point of λ. Then there exists $\nu \in \mathcal{R}(x_t, r, s)$ such that $\lambda \leq \underline{1} - \nu$. Since, $\underline{1} - \nu = \mathcal{C}(\mathcal{I}(\underline{1} - \nu, r, s), r, s)$, we have $\mathcal{D}(\lambda, r, s) \leq \underline{1} - \nu$. Furthermore, $x_t q \nu$ implies $\mathcal{D}(\lambda, r, s)(x) \leq (\underline{1} - \nu)(x) < t$. Hence $x_t \notin \mathcal{D}(\lambda, r, s)$.

Others are similarly proved.

(5) Since $\mathcal{R}(x_t, r, s) \subset \mathcal{Q}(x_t, r, s)$, then $x_t \in \mathcal{C}(\lambda, r, s)$ implies $x_t \in \mathcal{D}(\lambda, r, s)$. Hence $\mathcal{C}(\lambda, r, s) \leq \mathcal{D}(\lambda, r, s)$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

Suppose there exist $\lambda \in I^X$, $t \in (0,1)$, $r \in I_0$ and $s \in I_1$ such that

$$\mathcal{D}(\lambda, r, s)(x) > t > T(\lambda, r, s)(x).$$

Since x_t is not (r,s)-fuzzy θ-cluster point of λ, there exists $\mu \in \mathcal{Q}(x_t, r, s)$ such that $\lambda \leq \underline{1} - \mathcal{C}(\mu, r, s)$. It implies $\lambda \leq \underline{1} - \mathcal{C}(\mu, r, s) \leq \underline{1} - \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)$. Since $\tau(\mu) \geq r$, $\tau^*(\mu) \leq s$ and $\mu \leq \mathcal{C}(\mu, r, s)$, we have $\mu = \mathcal{I}(\mu, r, s) \leq \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)$. Thus, $x_t q \mu$ implies $x_t q \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)$. Since $\mathcal{I}(\mathcal{C}(\mu, r, s), r, s) \in \mathcal{R}(x_t, r, s)$ and $\lambda \leq \underline{1} - \mathcal{I}(\mathcal{C}(\mu, r, s), r, s)$. Hence x_t is not (r,s)-fuzzy δ-cluster point of λ. By (1), $\mathcal{D}(\lambda, r, s)(x) < t$. It is a contradiction. Hence $\mathcal{D}(\lambda, r, s) \leq T(\lambda, r, s)$ for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

Theorem 2.4 Let (X, τ, τ^*) be an ifts. For $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$, the operator $\mathcal{D}$ satisfies the following conditions:

(1) $\mathcal{D}(\underline{0}, r, s) = \underline{0}$.

(2) $\lambda \leq \mathcal{D}(\lambda, r, s)$.

(3) $\mathcal{D}(\lambda, r, s) \leq \mathcal{D}(\mu, r, s)$ if $\lambda \leq \mu$.

(4) $\mathcal{D}(\lambda, r, s) \vee \mathcal{D}(\mu, r, s) = \mathcal{D}(\lambda \vee \mu, r, s)$.

(5) $\mathcal{D}(\mathcal{D}(\lambda, r, s), r, s) = \mathcal{D}(\lambda, r, s)$.

Proof. (1),(2) and (3) are easily proved from the definition of $\mathcal{D}$.

(4) From (3) we have $\mathcal{D}(\lambda, r, s) \vee \mathcal{D}(\mu, r, s) \leq \mathcal{D}(\lambda \vee \mu, r, s)$.

Suppose there exist $\lambda_1, \lambda_2 \in I^X$, $x \in X$, $r \in I_0, s \in I_1$ and $t \in (0,1)$ such that

$$\mathcal{D}(\lambda_1, r, s)(x) \vee \mathcal{D}(\lambda_2, r, s)(x) < t \leq \mathcal{D}(\lambda_1 \vee \lambda_2, r, s)(x). \tag{1}$$

For each $i \in \{1,2\}$, since $\mathcal{D}(\lambda_i, r, s)(x) < t$, by Theorem 2.3(2), there exists $\nu_i \in \mathcal{R}(x_t, r, s)$ with $\lambda_i \leq \nu_i = \mathcal{C}(\mathcal{I}(\nu_i, r, s), r, s)$ such that

$$\mathcal{D}(\lambda_1, r, s)(x) \vee \mathcal{D}(\lambda_2, r, s)(x) \leq (\nu_1 \vee \nu_2, r, s)(x) < t. \tag{2}$$

Since $\nu_i = \mathcal{C}(\mathcal{I}(\nu_i, r, s), r, s)$, we have

$$\mathcal{C}(\mathcal{I}(\nu_1 \vee \nu_2, r, s), r, s) \leq \mathcal{C}(\nu_1 \vee \nu_2, r, s) = \mathcal{C}(\nu_1, r, s) \vee \mathcal{C}(\nu_2, r, s) = \nu_1 \vee \nu_2.$$

Moreover, $\mathcal{C}(\mathcal{I}(\nu_1 \vee \nu_2, r, s), r, s) \geq \mathcal{C}(\mathcal{I}(\nu_1, r, s), r, s) \vee \mathcal{C}(\mathcal{I}(\nu_2, r, s), r, s) = \nu_1 \vee \nu_2$. Thus, $\mathcal{C}(\mathcal{I}(\nu_1 \vee \nu_2, r, s), r, s) = \nu_1 \vee \nu_2$. Hence $\mathcal{D}(\lambda_1 \vee \lambda_2, r, s) \leq \nu_1 \vee \nu_2$. It is a contradiction for the equations (1) and (2).

(5) From (2), $\mathcal{D}(\mathcal{D}(\lambda, r, s), r, s) \geq \mathcal{D}(\lambda, r, s)$. Suppose

$$\mathcal{D}(\mathcal{D}(\lambda, r, s), r, s)(x) > t > \mathcal{D}(\lambda, r, s)(x).$$

Then there exists $\nu \in I^X$ with $\lambda \leq \nu = \mathcal{C}(\mathcal{I}(\nu, r, s), r, s)$, such that

$$\mathcal{D}(\mathcal{D}(\lambda, r, s), r, s)(x) > t > \nu(x) \geq \mathcal{D}(\lambda, r, s)(x).$$

Since $\lambda \leq \nu = \mathcal{C}(\mathcal{I}(\nu, r, s), r, s)$, we have

$$\mathcal{D}(\lambda, r, s) \leq \mathcal{D}(\nu, r, s) = \mathcal{D}(\mathcal{C}(\mathcal{I}(\nu, r, s), r, s), r, s) = \mathcal{C}(\mathcal{I}(\nu, r, s), r, s), r, s) = \nu.$$

Thus, $\mathcal{D}(\mathcal{D}(\lambda, r, s), r, s) \leq \nu$. It is a contradiction.

Theorem 2.5 Let (X, τ, τ^*) be an ifts. For $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$, the operator T satisfies the following conditions:

(1) $T(\underline{0}, r, s) = \underline{0}$.

(2) $\lambda \leq T(\lambda, r, s)$.

(3) $T(\lambda, r, s) \leq T(\mu, r, s)$ if $\lambda \leq \mu$.

(4) $T(\lambda, r, s) \vee T(\mu, r, s) = T(\lambda \vee \mu, r, s)$.

Theorem 2.6 Let (X, τ, τ^*) be an ifts. For $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. Define the mappings $\tau_\theta, \tau^*_\theta, \tau_\delta, \tau^*_\delta : I^X \to I$ by

$$\tau_\theta(\lambda) = \bigvee \{r \in I_0 \mid T(\underline{1} - \lambda, r_0, s_0) = \underline{1} - \lambda, \quad 0 < r_0 \leq r, \quad s \leq s_0 < 1\},$$

$$\tau^*_\theta(\lambda) = \bigwedge \{s \in I_1 \mid T(\underline{1} - \lambda, r_0, s_0) = \underline{1} - \lambda, \quad 0 < r_0 \leq r, \quad s \leq s_0 < 1\},$$

$$\tau_\delta(\lambda) = \bigvee \{r \in I_0 \mid \mathcal{D}(\underline{1} - \lambda, r_0, s_0) = \underline{1} - \lambda, \quad 0 < r_0 \leq r, \quad s \leq s_0 < 1\},$$

$$\tau^*_\theta(\lambda) = \bigwedge \{s \in I_1 \mid \mathcal{D}(\underline{1} - \lambda, r_0, s_0) = \underline{1} - \lambda, \quad 0 < r_0 \leq r, \quad s \leq s_0 < 1\}.$$

(1) $\tau_\theta \leq \tau_\delta \leq \tau$ and $\tau^*_\theta \geq \tau^*_\delta \geq \tau^*$.

(2) $(\tau_\theta, \tau^*_\theta)$ and $(\tau_\delta, \tau^*_\delta)$ are IGO's on X.

Proof. (1) We will show that $\tau_\theta \leq \tau_\delta$ and $\tau^*_\theta \geq \tau^*_\delta$. Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\tau_\theta(\lambda) > r > \tau_\delta(\lambda)$ and $\tau^*_\theta(\lambda) < s < \tau^*_\delta(\lambda)$. Then there exists $r_0 \in I_0$ and $s_0 \in I_1$ with $0 < r^* \leq r_0$, $s_0 \leq s^* < 1$, $\underline{1} - \lambda = T(\underline{1} - \lambda, r^*, s^*)$ such that $\tau_\theta(\lambda) \geq r_0 > r$ and $\tau^*_\theta(\lambda) \leq s_0 < s$. On the other hand, since $T(\underline{1} - \lambda, r^*, s^*) \geq \mathcal{D}(\underline{1} - \lambda, r^*, s^*)$ for each

$0 < r^* \leq r_0$, $s_0 \leq s^* < 1$, we have $\underline{1} - \lambda = \mathcal{D}(\underline{1} - \lambda, r^*, s^*)$. Thus, $\tau_\delta(\lambda) \geq r_0$ and $\tau_\delta^*(\lambda) \leq s_0$. It is a contradiction.

(2) First, we will show that $(\tau_\theta, \tau_\theta^*)$ is an IGO on X.

(IGO1) It is easy prove by (1) and Definition 1.1.

(IGO2) For all $r \in I_0$ and $s \in I_1$, we have $T(\underline{0}, r, s) = \underline{0}$ and $T(\underline{1}, r, s) = \underline{1}$. Hence $\tau_\theta(\underline{0}) = \tau_\theta(\underline{1}) = 1$ and $\tau_\theta^*(\underline{0}) = \tau_\theta^*(\underline{1}) = 0$.

(IGO3) Suppose there exist $\lambda_1, \lambda_2 \in I^X$ and $t, m \in (0, 1)$ such that

$$\tau_\theta(\lambda_1 \wedge \lambda_2) < t < \tau_\theta(\lambda_1) \wedge \tau_\theta(\lambda_2).$$

$$\tau_\theta^*(\lambda_1 \wedge \lambda_2) > m > \tau_\theta^*(\lambda_1) \vee \tau_\theta^*(\lambda_2).$$

From the definition of $(\tau_\theta, \tau_\theta^*)$, there exist $r_i \in I_0$, $s_i \in I_1$ for $i \in \{1, 2\}$ with $t \leq r_i, m \geq s_i$ and for each $0 < r_0 \leq r_i$, $s_i < s_0 < 1$,

$$\underline{1} - \lambda_i = T(\underline{1} - \lambda_i, r_0, s_0),$$

such that

$$\tau_\theta(\lambda_1 \wedge \lambda_2) < r_1 \wedge r_2 < \tau_\theta(\lambda_1) \wedge \tau_\theta(\lambda_2),$$
$$\tau_\theta^*(\lambda_1 \wedge \lambda_2) > s_1 \vee s_2 \geq \tau_\theta^*(\lambda_1) \vee \tau_\theta^*(\lambda_2).$$

Put $r_1 \wedge r_2 = r^*, s_1 \vee s_2 = s^*$. Since $\underline{1} - \lambda_i = T(\underline{1} - \lambda_i, r_0, s_0)$, for all $0 < r_0 \leq r_i$, $s_i < s_0 < 1$,

$$\begin{aligned} T(\underline{1} - (\lambda_1 \wedge \lambda_2), r_0, s_0) &= T(\underline{1} - \lambda_1) \vee (\underline{1} - \lambda_2), r_0, s_0) \\ &= T(\underline{1} - \lambda_1, r_0, s_0) \vee T(\underline{1} - \lambda_2, r_0, s_0) \\ &= (\underline{1} - \lambda_1) \vee (\underline{1} - \lambda_2). \end{aligned}$$

Thus, $\tau_\theta(\lambda_1 \wedge \lambda_2) \geq r_0$ and $\tau_\theta^*(\lambda_1 \wedge \lambda_2) \leq s_0$. It is a contradiction.

(IGO4) Suppose there exist $\{\lambda_j \in I^X\}_{j \in J}$ and $t, m \in (0, 1)$ such that

$$\tau_\theta(\bigvee_{j \in J} \lambda_j) < t < \bigwedge_{j \in J} \tau_\theta(\lambda_j) \text{ and } \tau_\theta^*(\bigvee_{j \in J} \lambda_j) > m > \bigvee_{j \in J} \tau_\theta^*(\lambda_j).$$

From the definition of $(\tau_\theta, \tau_\theta^*)$, there exist $r_j \in I_0$, $s_j \in I_1$ for $j \in J$ with $0 < r_0 \leq r_j$, $s_j \leq s_0 < 1$, $\underline{1} - \lambda_j = T(\underline{1} - \lambda_j, r_0, s_0)$ such that

$$\tau_\theta(\bigvee_{j \in J} \lambda_j) < t < \bigwedge_{j \in J} r_j < \bigwedge_{j \in J} \tau_\theta(\lambda_j),$$

$$\tau_\theta^*(\bigvee_{j \in J} \lambda_j) > m > \bigvee_{j \in J} s_j \geq \bigvee_{j \in J} \tau_\theta^*(\lambda_j).$$

Put $r^* = \bigwedge_{j \in J} r_j$, $s^* = \bigvee_{j \in J} s_j$. Then

$$\begin{aligned} T(\underline{1} - (\bigvee_{j \in J} \lambda_j), r_0, s_0) &= T(\bigwedge_{j \in J} (\underline{1} - \lambda_j), r_0, s_0) \\ &\leq \bigwedge_{j \in J} T(\underline{1} - \lambda_j), r_0, s_0) = \bigwedge_{j \in J} (\underline{1} - \lambda_j) = \underline{1} - \bigvee_{j \in J} \lambda_j. \end{aligned}$$

Hence, $T(\underline{1} - \bigvee_{j \in J} \lambda_j, r_0, s_0) = \underline{1} - \bigvee_{j \in J} \lambda_j$. Thus, $\tau_\theta(\bigvee_{j \in J} \lambda_j) \geq r_0$ and $\tau_\theta^*(\bigvee_{j \in J} \lambda_j) \leq s_0$. It is a contradiction.

3 Fuzzy δ-connectedness and fuzzy θ-connectedness

Definition 3.1 Let (X,τ,τ^*) be an ifts. For each $\lambda\in I^X$, $r\in I_\circ$ and $s\in I_1$.

(1) λ is called (r,s)-fuzzy θ-closed (resp. (r,s)-fuzzy δ-closed) iff $\lambda=T(\lambda,r,s)$ (resp. $\lambda=\mathcal{D}(\lambda,r,s)$). We define

$$\Delta(\lambda,r,s)=\bigwedge\{\mu\in I^X:\lambda\le\mu,\ \mu=\mathcal{D}(\mu,r,s)\},$$

$$\Theta(\lambda,r,s)=\bigwedge\{\mu\in I^X:\lambda\le\mu,\ \mu=T(\mu,r,s)\}.$$

(2) The complement of (r,s)-fuzzy θ-closed (resp. (r,s)-fuzzy δ-closed) set is called (r,s)-fuzzy θ-open (resp. (r,s)-fuzzy δ-open).

Theorem 3.2 Let (X,τ,τ^*) be an ifts. For each $\lambda\in I^X$, $r\in I_\circ$ and $s\in I_1$, we have the following properties:

(1) $\Delta(\lambda,r,s)=\mathcal{D}(\lambda,r,s)$.

(2) $\Delta(\lambda,r,s)$ is (r,s)-fuzzy δ-closed.

(3) $\Theta(\lambda,r,s)=T(\Theta(\lambda,r,s),r,s)$,i.e., $\Theta(\lambda,r,s)$ is (r,s)-fuzzy θ-closed.

(4) $T(\lambda,r,s)\le\Theta(\lambda,r,s)$.

Proof. (1) Since $\lambda\le\mathcal{D}(\lambda,r,s)=\mathcal{D}(\mathcal{D}(\lambda,r,s),r,s)$, we have $\Delta(\lambda,r,s)\le\mathcal{D}(\lambda,r,s)$.

Suppose $\Delta(\lambda,r,s)\not\ge\mathcal{D}(\lambda,r,s)$. There exist $x\in X$ and $t\in(0,1)$ such that

$$\Delta(\lambda,r,s)(x)<t<\mathcal{D}(\lambda,r,s)(x).$$

From the definition of $\Delta(\lambda,r,s)$, there exists $\mu\in I^X$ with $\lambda\le\mu=\mathcal{D}(\mu,r,s)$ such that

$$\Delta(\lambda,r,s)(x)\le\mu(x)<t<\mathcal{D}(\lambda,r,s)(x).$$

On the other hand, since $\lambda\le\mu$, we have $\mathcal{D}(\lambda,r,s)\le\mathcal{D}(\mu,r,s)=\mu$. It is a contradiction. Hence $\Delta(\lambda,r,s)\ge\mathcal{D}(\lambda,r,s)$.

(2) It is trivial.

(3) Let $\lambda\le\mu_j=T(\mu_j,r,s)$ for each $j\in\Gamma$. Then

$$\bigwedge_{j\in\Gamma}\mu_j\le T(\bigwedge_{j\in\Gamma}\mu_j,r,s)\le T(\mu_j,r,s)=\mu_j.$$

It implies $\bigwedge_{j\in\Gamma}\mu_j=T(\bigwedge_{j\in\Gamma}\mu_j,r,s)$. Hence $\Theta(\lambda,r,s)=T(\Theta(\lambda,r,s),r,s)$, that is, $\Theta(\lambda,r,s)$ is (r,s)-fuzzy θ-closed set.

(4) Since $\lambda\le\Theta(\lambda,r,s)$, by (3), we have $T(\lambda,r,s)\le T(\Theta(\lambda,r,s),r,s)=\Theta(\lambda,r,s)$.

In general, by Theorem 3.2(1-2), an $\delta-$ closure operator is (r,s)-fuzzy δ-closed for each $r\in I_0$ and $s\in I_1$, but an $\theta-$ closure operator is not (r,s)-fuzzy-θ-closed.

Example 3.3 Let $X=\{x,y\}$ be a set. Let (X,τ,τ^*) be an ifts as follows:

$$\tau(\lambda)=\begin{cases}1, & \lambda\in\{\underline{0},\underline{1}\}\\ \frac{1}{2}, & \lambda=\underline{0.7},\\ \frac{1}{2}, & \lambda=\underline{0.5},\\ \frac{1}{2}, & \lambda=\underline{0.4},\\ 0, & \text{otherwise},\end{cases}\qquad \tau^*(\lambda)=\begin{cases}0, & \lambda\in\{\underline{0},\underline{1}\}\\ \frac{1}{2}, & \lambda=\underline{0.7},\\ \frac{1}{2}, & \lambda=\underline{0.5},\\ \frac{1}{2}, & \lambda=\underline{0.4},\\ 1, & \text{otherwise}.\end{cases}$$

We obtain

$$T(\lambda, r, s) = \begin{cases} \underline{0}, & \lambda = \underline{0},\ r \in I_0, s \in I_1 \\ \underline{0.6}, & \underline{0} \neq \lambda \leq \underline{0.5},\ 0 < r \leq \frac{1}{2}, \frac{1}{2} \leq s < 1, \\ \underline{1}, & \text{otherwise.} \end{cases}$$

Since

$$\underline{1} = T(T(\underline{0.5}, \frac{1}{2}, \frac{1}{2}), \frac{1}{2}, \frac{1}{2}) \neq T(\underline{0.5}, \frac{1}{2}, \frac{1}{2}) = \underline{0.6},$$

then $T(\underline{0.5}, \frac{1}{2}, \frac{1}{2})$ is not $(\frac{1}{2}, \frac{1}{2})$-fuzzy $\theta-$closed. Since

$$\Theta(\lambda, r, s) = \begin{cases} \underline{0}, & \lambda = \underline{0},\ r \in I_0, s \in I_1 \\ \underline{1}, & \text{otherwise,} \end{cases}$$

we have $T(\lambda, r, s) \leq \Theta(\lambda, r, s)$.

Definition 3.4 A pair (λ, μ) of non-null fuzzy sets in an ifts (X, τ, τ^*) is said to be (r, s)-fuzzy separation relative to X, $r \in I_0$ and $s \in I_1$ iff $\lambda \overline{q} \mu$, $\lambda \overline{q} \mathcal{C}(\mu, r, s)$ and $\mathcal{C}(\lambda, r, s) \overline{q} \mu$. A fuzzy set γ in an ifts X is said to be (r, s)-fuzzy connected iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an (r, s)-fuzzy separation relative to X and $\gamma = \lambda \vee \mu$.

Definition 3.5 A pair (λ, μ) of non-null fuzzy sets in an ifts (X, τ, τ^*), $r \in I_0$ and $s \in I_1$ is said to be (r, s)-fuzzy-θ-separation relative to X iff $\lambda \overline{q} \mu$, $\lambda \overline{q} \Theta(\mu, r, s)$ and $\Theta(\lambda, r, s) \overline{q} \mu$. A fuzzy set γ in an ifts X is said to be (r, s)-fuzzy-θ-connected iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an (r, s)-fuzzy-θ-separation relative to X and $\gamma = \lambda \vee \mu$.

Definition 3.6 A pair (λ, μ) of non-null fuzzy sets in an ifts (X, τ, τ^*) is said to be (r, s)-fuzzy-δ-separation relative to X iff $\lambda \overline{q} \mu$, $\lambda \overline{q} \Delta(\mu, r, s)$ and $\Delta(\lambda, r, s) \overline{q} \mu$. A fuzzy set γ in an ifts X is said to be (r, s)-fuzzy-δ-connected iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an (r, s)-fuzzy-δ-separation relative to X and $\gamma = \lambda \vee \mu$.

Remark 3.7 From Theorem 3.2(1,4), it is clear that:
(r, s)-fuzzy connected $\Rightarrow$ (r, s)-fuzzy-δ-connected $\Rightarrow$ (r, s)-fuzzy-θ-connected.

Example 3.8 Let $X = I$ and (X, τ, τ^*) an ifts define as

$$\tau(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \lambda = \lambda_1, \lambda_2, \\ \frac{1}{3}, & \lambda = \lambda_3, \lambda_4, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \lambda = \lambda_1, \lambda_2, \\ \frac{2}{3}, & \lambda = \lambda_3, \lambda_4, \\ 1, & \text{otherwise,} \end{cases}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are fuzzy sets defined as follows

$$\lambda_1(0) = \frac{1}{4}, \quad \lambda_2(0) = \frac{1}{7}, \quad \lambda_3(0) = \frac{8}{9}, \quad \lambda_4(0) = \frac{1}{5} \text{ and}$$

$$\lambda_k(x) = \frac{1}{2} \quad \forall x \in I_0, k = 1, 2, 3, 4.$$

Then (X, τ, τ^*) is an ifts. Consider a fuzzy set defined as follows

$$\gamma(x) = \begin{cases} \frac{6}{7}, & x = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Then, $\gamma = \mu \vee \nu$, where

$$\mu(x) = \begin{cases} \frac{6}{7}, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \quad \nu(x) = \begin{cases} \frac{1}{10}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Obviously, $\mu \bar{q} \nu$. Again

$$\mathcal{C}(\mu, \frac{1}{3}, \frac{2}{3}) = \begin{cases} \frac{6}{7}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases} \qquad \text{and} \quad \mathcal{C}(\nu, \frac{1}{3}) = \begin{cases} \frac{1}{9}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Obviously, $\mu \bar{q} \mathcal{C}(\nu, \frac{1}{3}, \frac{2}{3})$ and $\mathcal{C}(\mu, \frac{1}{3}, \frac{2}{3}) \bar{q} \nu$. Then γ is not $(\frac{1}{3}, \frac{2}{3})$-fuzzy connected. For any respresentation $\gamma = \mu \vee \nu$, where μ and ν are non-empty, either $\mu(0) = \frac{6}{7}$ or $\nu(0) = \frac{6}{7}$, by Theorem 3.2(1), then if $\mu(0) = \frac{6}{7}$ (resp. if $\nu(0) = \frac{6}{7}$), then $\Delta(\mu, \frac{1}{3}, \frac{2}{3}) = \underline{1}$ (resp. $\Delta(\nu, \frac{1}{3}, \frac{2}{3}) = \underline{1}$) so that γ is not representable as $\mu \vee \nu$, where (μ, ν) is an $(\frac{1}{3}, \frac{2}{3})$-fuzzy δ-separation. Hence γ is $(\frac{1}{3}, \frac{2}{3})$-fuzzy δ-connected.

Example 3.9 Define an ifts (X, τ, τ^*) as in Example 3.3. From Theorem 3.2(1), we obtain

$$\Delta(\lambda, r, s) = \begin{cases} \underline{0}, & \lambda = \underline{0},\ r \in I_0, s \in I_1, \\ \underline{0.6}, & \underline{0} \neq \lambda \leq \underline{0.6},\ 0 < r \leq \frac{1}{2}, \frac{1}{2} \leq s < 1 \\ \underline{1}, & \text{otherwise}. \end{cases}$$

For $\underline{0.4} = \underline{0.3} \vee \underline{0.4}$, we have $\underline{0.3}\ \bar{q}\ \underline{0.4}$, $\underline{0.6} = \Delta(\underline{0.3}, \frac{1}{3}, \frac{2}{3})\ \bar{q}\ \underline{0.4}$, $\underline{0.3} \bar{q} \Delta(\underline{0.4}, \frac{1}{3}, \frac{2}{3}) = \underline{0.6}$. Hence $(\underline{0.3}, \underline{0.4})$ is an $(\frac{1}{3}, \frac{2}{3})$-fuzzy-$\delta$-separation and $\underline{0.4}$ is not $(\frac{1}{3}, \frac{2}{3})$-fuzzy-$\delta$-connected.

For any representation $\underline{0.4} = \mu \vee \nu$, where μ and ν are non-empty, by Example 3.3, $\Theta(\lambda, r, s) = \underline{1}$ for $\lambda \in \{\mu, \nu\}$. Thus, $\underline{0.4}$ is $(\frac{1}{3}, \frac{2}{3})$-fuzzy-$\theta$-connected.

Definition 3.10 Let (X, τ, τ^*) be an ifts, $r \in I_0$ and $s \in I_1$. Then X is said to be

(1) (r, s)-fuzzy regular iff for each fuzzy point x_t in X and each $\mu \in \mathcal{Q}(x_t, r, s)$ there exists $\nu \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{C}(\nu, r, s) \leq \mu$.

(2) (r, s)-fuzzy almost regular iff for each fuzzy point x_t in X and each $\mu \in \mathcal{R}(x_t, r, s)$ there exists $\nu \in \mathcal{R}(x_t, r, s)$ such that $\mathcal{C}(\nu, r, s) \leq \mu$.

(3) (r, s)-fuzzy semi-regular iff for each $\mu \in \mathcal{Q}(x_t, r, s)$, there exists $\rho \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \leq \mu$.

Theorem 3.11 Let (X, τ, τ^*) be an ifts, $r \in I_0$ and $s \in I_1$. Then

(1) X is (r, s)-fuzzy almost regular iff $T(\lambda, r, s) = \mathcal{D}(\lambda, r, s)$, for each $\lambda \in I^X$.

(2) X is (r, s)-fuzzy semi-regular iff $\mathcal{D}(\lambda, r, s) = \mathcal{C}(\lambda, r, s)$, for each $\lambda \in I^X$.

Proof. (1) We only show that $T(\lambda, r, s) \leq \mathcal{D}(\lambda, r, s)$. Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that $T(\lambda, r, s) \not\leq \mathcal{D}(\lambda, r, s)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $T(\lambda, r, s)(x) > t > \mathcal{D}(\lambda, r, s)(x)$. Since $\mathcal{D}(\lambda, r, s)(x) < t$, x_t is not (r, s)-fuzzy δ-cluster point of λ. Then there exists $\mu \in \mathcal{R}(x_t, r, s)$ such that $\lambda \leq \underline{1} - \mu = \mathcal{C}(\mathcal{I}(\underline{1} - \mu, r, s), r, s)$. Since (X, τ, τ^*) is (r, s)-fuzzy almost regular, for $\mu \in \mathcal{R}(x_t, r, s)$, there exists $\rho \in \mathcal{R}(x_t, r, s)$ such that $\mathcal{C}(\rho, r, s) \leq \mu$. Thus

$$\lambda \leq \underline{1} - \mu \leq \underline{1} - \mathcal{C}(\rho, r, s) \leq \mathcal{I}(\underline{1} - \rho, r, s) \leq \mathcal{I}(\mathcal{C}(\mathcal{I}(\underline{1} - \rho, r, s), r, s), r, s).$$

It follows $T(\lambda, r, s) \leq \mathcal{C}(\mathcal{I}(\underline{1} - \rho, r, s), r, s)$. Hence $T(\lambda, r, s)(x) \leq (\underline{1} - \rho)(x) < t$. It is a contradiction.

Conversely, for each $\mu \in \mathcal{R}(x_t, r, s)$, $t > (\underline{1} - \mu)(x) = \mathcal{C}(\mathcal{I}(\underline{1} - \mu, r, s), r, s)(x)$. Since

$$\begin{aligned} T(\underline{1} - \mu, r) &= T(\mathcal{C}(\mathcal{I}(\underline{1} - \mu, r, s), r, s), r, s) \\ &= \mathcal{D}(\mathcal{C}(\mathcal{I}(\underline{1} - \mu, r, s), r, s), r, s) \\ &= \mathcal{C}(\mathcal{I}(\underline{1} - \mu, r, s), r, s), \end{aligned}$$

then x_t is not (r, s)-fuzzy θ-cluster point of $\underline{1} - \mu$. Then there exists $\rho \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{C}(\rho, r, s) \leq \mu$. It implies $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \leq \mathcal{C}(\mathcal{I}(\mathcal{C}(\rho, r, s), r, s), r, s) \leq \mu$. Moreover, $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \in \mathcal{Q}(x_t, r, s)$. Hence (X, τ, τ^*) is (r, s)-fuzzy almost regular.

(2) Similarly prove as (1).

Corollary 3.12 (1) In an (r, s)-fuzzy almost regular space, since, by Theorem 3.11(1), $\Theta(\lambda, r, s) = \Delta(\lambda, r, s)$, for each $\lambda \in I^X$, $r \in I_\circ$ and $s \in I_1$, (r, s)-fuzzy -δ-connectedness and (r, s)-fuzzy-θ-connectedness are equivalent.

(2) In an (r, s)-fuzzy semi-regular space, the concepts of (r, s)-fuzzy connectedness and that of (r, s)-fuzzy-δ-connectedness are equivalent.

Theorem 3.13 An ifts (X, τ, τ^*) is (r, s)-fuzzy regular iff it is (r, s)-fuzzy almost regular and (r, s)-fuzzy semi-regular.

Proof. Let $\mu \in \mathcal{Q}(x_t, r, s)$. Since (X, τ, τ^*) is (r, s)-fuzzy regular, there exists $\rho \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{C}(\rho, r, s) \leq \mu$. So, $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \leq \mathcal{C}(\rho, r, s) \leq \mu$. Hence (X, τ, τ^*) is (r, s)-fuzzy semi-regular. Let $\mu \in \mathcal{R}(x_t, r, s)$. Since (X, τ, τ^*) is (r, s)-fuzzy regular and $\mu \in \mathcal{Q}(x_t, r, s)$, there exists $\rho \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{C}(\rho, r, s) \leq \mu$. Since $\rho \leq \mathcal{I}(\mathcal{C}(\rho, r, s), r, s)$, $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \in \mathcal{R}(x_t, r, s)$. So, $\mathcal{C}(\mathcal{I}(\mathcal{C}(\rho, r, s), r, s), r, s) = \mathcal{C}(\rho, r, s) \leq \mu$. Hence (X, τ, τ^*) is (r, s)-fuzzy almost regular.

Conversely, for each $\mu \in \mathcal{Q}(x_t, r, s)$, since (X, τ, τ^*) is (r, s)-fuzzy semi-regular, there exists $\rho \in \mathcal{Q}(x_t, r, s)$ such that $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \leq \mu$. It follows $\mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \in \mathcal{R}(x_t, r, s)$. Since (X, τ, τ^*) is (r, s)-fuzzy almost regular, there exists $\nu \in \mathcal{R}(x_t, r, s) \subset \mathcal{Q}(x_t, r, s)$ such that $\mathcal{C}(\nu, r, s) \leq \mathcal{I}(\mathcal{C}(\rho, r, s), r, s) \leq \mu$. Hence (X, τ, τ^*) is (r, s)-fuzzy regular.

Corollary 3.14 In an (r, s)-fuzzy regular space, since, by Theorems 3.2, 3.11(1,2) and 3.12, $\mathcal{C}(\lambda, r, s) = \Theta(\lambda, r, s) = \Delta(\lambda, r, s)$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, then the notions of (r, s)-fuzzy connectedness, (r, s)-fuzzy-δ-connectedness and (r, s)-fuzzy-θ-connectedness of fuzzy sets become identical.

References

[1] S. E. Abbas, (r, s)-generalized intuitionistic fuzzy closed sets, J. Egypt Math. Soc. vol 14(2), 2006, 283-297

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, vol 20(1), 1986, 87-96

[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., vol 24, 1968, 182-190

[4] K. C. Chattopadhyay, R. N. Hazra, S. K. Samanta, Gradation of openess: fuzzy topology, Fuzzy Sets and Systems, vol 49, 1992, 237-242

[5] Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, vol 88, 1997, 81-89

[6] D. Çoker and M. Dimirci, An introduction to intuitionistic fuzzy topological spaces in Šostak sense, Buseful, vol 67, 1996, 67-76

[7] U. Höhle, Upper semicontinuous fuzzy sets and applications, J. Math. Anall. Appl., vol 78, 1980, 659-673

[8] U. Höhle and A. P. Šostak, A general theory of fuzzy topological spaces, Fuzzy Sets and Systems, vol 73, 1995, 131-149

[9] U. Höhle and A. P. Šostak, Axiomatic Foundations of Fixed-Basis fuzzy topology, The Handbooks of Fuzzy sets series, Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3), 1999

[10] T. Kubiak, On fuzzy topologies, Ph. D. Thesis, A. Mickiewicz, Poznan, 1985

[11] T. Kubiak and A. P. Šostak, Lower set-valued fuzzy topologies, Quaestions Math., vol 20(3), 1997, 423-429

[12] E. P. Le and Y. B. Im, Mated fuzzy topological spaces, J. Korea Fuzzy Logic Intell. Sys. Soc., vol 11(2), 2001, 161-165

[13] S. K. Samanta and T. K. Mondal, On intuitionistic gradation of openness, Fuzzy Sets and Systems, vol 131, 2002, 323-336

[14] A. P. Šostak, On a fuzzy topological structure, Suppl. Rend. Circ. Matem. Palerms ser II, vol 11, 1985, 89-103