Some new fixed point theorems
in generalized intuitionistic fuzzy metric spaces

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This paper is dedicated to our advisors.

Abstract: The aim of this paper is to give some new fixed point theorems for contractive type mappings in generalized intuitionistic fuzzy metric spaces. The results presented improve some well known results in the literature.
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1 Introduction

The concept of fuzzy metric space was introduced initially by Kramosil and Michalek [6]. Later on, George and Veeramani [4] give the modified notion of fuzzy metric spaces due to Kramosil and Michalek and analyzed a Hausdorff topology of fuzzy metric spaces. In 2006, Sedghi and Shobe [8] defined $\mathcal{M}$-fuzzy metric spaces and proved a common fixed point theorem for weakly compatible mapping in this space.

On the other hand, the concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of fuzzy set. In 2004, Park [7] introduced the notion of intuitionistic fuzzy metric space. In this paper, we give some new fixed point theorems in generalized intuitionistic fuzzy metric spaces.

2 Preliminaries

Definition 2.1. The 5-tuble $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is a generalized intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and $\mathcal{M}, \mathcal{N}$ are fuzzy sets on $X^3 \times [0, 1)$ satisfying the following conditions: for all $x, y, z \in X$,

   (i) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$,

  (ii) $\mathcal{M}(x, y, z, 0) = 0$,

 (iii) $\mathcal{M}(x, y, z, t) = 1$ for all $t > 0$ iff $x = y = z$,

 (iv) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, when p is the permutation function,

  (v) $\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s)$ for all $t, s > 0$,

 (vi) $\mathcal{M}(x, y, z, .) : [0, 1) \to [0, 1]$ is left-continuous,

(vii) $\lim\limits_{t \to \infty} \mathcal{M}(x, y, z, t) = 1$,

(viii) $\mathcal{N}(x, y, z, 0) = 1$,

 (ix) $\mathcal{N}(x, y, z, t) = 0$ for all $t > 0$ iff $x = y = z$,

  (x) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$, when p is the permutation function,

 (xi) $\mathcal{N}(x, y, z, t + s) \leq \mathcal{N}(x, y, a, t) \diamond \mathcal{N}(a, z, z, s)$ for all $t, s > 0$,

(xii) $\mathcal{N}(x, y, z, .) : [0, 1) \to [0, 1]$ is right-continuous,

(xiii) $\lim\limits_{t \to \infty} \mathcal{N}(x, y, z, t) = 0$.

Remark 2.2. By (iii) and (v), it is easy to show that $\mathcal{M}(x, y, z, .)$ is non-decreasing and by (ix) and (xi), it is easy to show that $\mathcal{N}(x, y, z, .)$ is non-increasing for all $x, y, z \in X$.

Remark 2.3. Every generalized fuzzy metric space $(X, \mathcal{M}, *)$ is a generalized intuitionistic fuzzy metric space of the form $(X, \mathcal{M}, 1 - \mathcal{M}, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e. $x \diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0, 1]$.

Definition 2.4. A sequence $\{x_n\}$ in a generalized intuitionistic fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is a Cauchy sequence iff

$$\lim\limits_{n \to \infty} \mathcal{M}(x_{n+p}, x_n, x_n, t) = 1 \quad \text{and} \quad \lim\limits_{n \to \infty} \mathcal{N}(x_{n+p}, x_n, x_n, t) = 0$$

for each $t > 0$ and $p \in \mathbb{N}$. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, x, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{N}(x_n, x, x, t) = 0$$

for each $t > 0$. A generalized intuitionistic fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ is called complete, if every Cauchy sequence in X is convergent. It is called compact, if every sequence in X contains a convergent subsequence.

Lemma 2.5. *Let* $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$ *and* $\lim\limits_{n\to\infty} z_n = z$. *Then, for all* $t > 0$,

$$\lim_{n\to\infty} \inf \mathcal{M}(x_n, y_n, z_n, t) \geq \mathcal{M}(x, y, z, t), \qquad \lim_{n\to\infty} \sup \mathcal{N}(x_n, y_n, z_n, t) \leq \mathcal{N}(x, y, z, t),$$

$$\lim_{n\to\infty} \sup \mathcal{M}(x_n, y_n, z_n, t) \leq \mathcal{M}(x, y, z, t), \qquad \lim_{n\to\infty} \inf \mathcal{N}(x_n, y_n, z_n, t) \geq \mathcal{N}(x, y, z, t)$$

Particularly, if $\mathcal{M}(x, y, z, .)$ *and* $\mathcal{N}(x, y, z, .)$ *are continuous at point* t, *then*

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t) = \mathcal{M}(x, y, z, t) \text{ and } \lim_{n\to\infty} \mathcal{N}(x_n, y_n, z_n, t) = \mathcal{N}(x, y, z, t)$$

3   Main results

Definition 3.1. Let the function $\phi : [0, \infty) \to [0, \infty)$ satisfy the following conditions:

(ϕ_1): $\phi(t)$ is strictly increasing, $\phi(0) = 0$ and $\lim\limits_{n\to\infty} \phi^n(t) = \infty$, for all $t > 0$, where $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$. Then $\phi(t) > t$ and $\phi^n(t) > \phi^{n-1}(t)$ for $t > 0$ and $n = 1, 2, \ldots$.

(ϕ_2): $\lim\limits_{t\to\infty} [\phi(t) - t] = \infty$.

Lemma 3.2. *Let* $*$ *and* $\diamond$ *be continuous* t-*norm and* t-*conorm, respectively. Then, for each* $\lambda \in (0, 1)$, *there is a sequence* $\{\lambda_n\}$ *in* $(0, 1)$ *such that*

$$(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1} \text{ and } \lambda_n \diamond \lambda_n < \lambda_{n-1}, n = 1, 2 \ldots,$$

where $\lambda_0 = \lambda$ *(obviously, the sequence* $\{\lambda_n\}$ *satisfying the condition is decreasing).*

Proof. Since $*$ is continuous at the point $(1, 1)$ and $a * b \leq 1 * 1 = 1$, and since $\diamond$ is continuous at the point $(0, 0)$ and $a \diamond b \geq 0 \diamond 0 = 0$ for all $a, b \in [0, 1]$, we have

$$\sup_{0 < \mu < 1} [(1 - \mu) * (1 - \mu)] = 1 \quad \text{and} \quad \inf_{0 < \mu < 1} \mu \diamond \mu = 0,$$

respectively. Thus, for each $\lambda \in (0, 1)$, there exists $\lambda_1 \in (0, 1)$ such that

$$(1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda \text{ and } \lambda_1 \diamond \lambda_1 < \lambda.$$

Similarly, there exists $\lambda_2 \in (0, 1)$ such that $(1 - \lambda_2) * (1 - \lambda_2) > 1 - \lambda_1$ and $\lambda_2 \diamond \lambda_2 < \lambda_1$. Continuing this procedure, we can obtain a sequence $\{\lambda_n\} \subset (0, 1)$ satisfying the condition. $\square$

Lemma 3.3. *Let* $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ *be a generalized intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a mapping satisfying*

$$\mathcal{M}(Tx, Ty, Tz, t_1) > \mathcal{M}(x, y, z, t_1) \quad and \quad \mathcal{N}(Tx, Ty, Tz, t_1) < \mathcal{N}(x, y, z, t_1),$$

where t_1 *is a fixed positive number. Then, there exists a continuity point* t_0 *of* $\mathcal{M}(x, y, z, .)$ *and* $\mathcal{N}(x, y, z, .)$ *such that*

$$\mathcal{M}(Tx, Ty, Tz, t_0) > \mathcal{M}(x, y, z, t_0) \text{ and } \mathcal{N}(Tx, Ty, Tz, t_0) < \mathcal{N}(x, y, z, t_0).$$

Proof. Since $\mathcal{M}(Tx, Ty, Tz, .) - \mathcal{M}(x, y, z, .)$ is left-continuous and $\mathcal{N}(Tx, Ty, Tz, .)$ $-\mathcal{N}(x, y, z, .)$ is right-continuous at point t_1, there exists $t_2 > 0$ such that

$$\mathcal{M}(Tx, Ty, Tz, t) > \mathcal{M}(x, y, z, t) \text{ and } \mathcal{N}(Tx, Ty, Tz, t) < \mathcal{N}(x, y, z, t)$$

for all $t \in [t_2, t_1]$. Note that the set of discontinuous points of $\mathcal{M}(x, y, z, .)$ and $\mathcal{N}(x, y, z, .)$ is countable atmost. Hence, there exists $t_0 \in [t_2, t_1]$ such that $\mathcal{M}(x, y, z, .)$ and $\mathcal{N}(x, y, z, .)$ are continuous at t_0. $\qquad\square$

Theorem 3.4. *Let* $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ *be a complete generalized intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a mapping satisfying the following conditions:*
(i) There exists $x_0 \in X$ *such that*

$$\lim_{t \to \infty} \mathcal{M}(x_0, T^i x_0, T^i x_0, t) = 1, \lim_{t \to \infty} \mathcal{N}(x_0, T^i x_0, T^i x_0, t) = 0, i = 1, 2, \ldots \qquad (3.4.1)$$

(ii) There exists a mapping $m : X \to \mathbb{N}$ *such that for any* $x, y, z \in X$

$$\mathcal{M}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \geq \mathcal{M}(x, y, z, \phi(t)) \quad and$$
$$\mathcal{N}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \leq \mathcal{N}(x, y, z, \phi(t)) \qquad (3.4.2)$$

where the function $\phi(t)$ *satisfies the conditions* (ϕ_1) *and* (ϕ_2).
Then T *has a unique fixed point* x^* *and the quasi-iterative sequence* $\{x_n = T^{m(x_{n-1})} x_{n-1}\}$ *converges to* x^*.

Proof. First, we prove that

$$\sup_{s > 0} \inf_{x \in O_T(x_0)} \mathcal{M}(x_0, x, x, s) = 1 \quad and \quad \inf_{s > 0} \sup_{x \in O_T(x_0)} \mathcal{N}(x_0, x, x, s) = 0, \qquad (3.4.3)$$

where $O_T(x_0) = \{x_0, Tx_0, T^2 x_0, \ldots\}$ is called the orbit of x_0 for T. For any $n \in \mathbb{N}$ with $n > m(x_0)$, we can denote $n = km(x_0) + s$, where $0 \leq s < m(x_0)$.
Note that $\phi(t) > t$ for all $t > 0$ and $\lim_{t \to \infty} [\phi(t) - t] = 1$.
By (3.4.1), we have

$$\lim_{t \to \infty} \mathcal{M}(x_0, T^i x_0, T^i x_0, \phi(t)) = 1 \quad and \quad \lim_{t \to \infty} \mathcal{N}(x_0, T^i x_0, T^i x_0, \phi(t)) = 0 \qquad (3.4.4)$$

for $i = 1, 2, \ldots, m(x_0)$ and

$$\lim_{t \to \infty} \mathcal{M}(x_0, T^i x_0, T^i x_0, \phi(t) - t) = 1 \quad and \quad \lim_{t \to \infty} \mathcal{N}(x_0, T^i x_0, T^i x_0, \phi(t) - t) = 0 \qquad (3.4.5)$$

Moreover, by Lemma 3.2, for any $\lambda \in (0,1)$, there is a sequence $\{\lambda_n\}$ in $(0,1)$ such that $(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1}$ and $\lambda_n \diamond \lambda_n < \lambda_{n-1}, n = 1, 2 \dots$.

Thus, it follows from (3.4.4) and (3.4.5) that for a given λ_k, there exists $t_0 > 0$ such that

$$\min_{1 \leq i \leq m(x_0)} \mathcal{M}(x_0, T^i x_0, T^i x_0, \phi(t)) > 1 - \lambda_k \quad \text{and}$$

$$\max_{1 \leq i \leq m(x_0)} \mathcal{N}(x_0, T^i x_0, T^i x_0, \phi(t)) < \lambda_k,$$

$$\mathcal{M}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) > 1 - \lambda_k \quad \text{and}$$

$$\mathcal{N}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) < \lambda_k$$

for all $t > t_0$. So, by (3.4.2), for all $t > t_0$, we have

$$\begin{aligned}
\mathcal{M}(x_0, T^n x_0, T^n x_0, \phi(t)) &= \mathcal{M}(x_0, T^{km(x_0)+s} x_0, T^{km(x_0)+s} x_0, \phi(t)) \\
&\geq \mathcal{M}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) * \mathcal{M}(T^{m(x_0)} x_0, T^{km(x_0)+s} x_0, T^{km(x_0)+s} x_0, t) \\
&\geq \mathcal{M}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) * \mathcal{M}(x_0, T^{(k-1)m(x_0)+s} x_0, T^{(k-1)m(x_0)+s} x_0, \phi(t)) \\
&\geq \cdots \geq \mathcal{M}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) * \dots^{(k)} * \mathcal{M}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) \\
&\quad * \mathcal{M}(x_0, T^s x_0, T^s x_0, \phi(t)) \\
&> (1 - \lambda_k) * \dots^{(k+1).} * (1 - \lambda_k) \\
&> (1 - \lambda_{k-1}) * \dots^{(k).} * (1 - \lambda_{k-1}) \\
&> \dots > (1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda.
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(x_0, T^n x_0, T^n x_0, \phi(t)) &= \mathcal{N}(x_0, T^{km(x_0)+s} x_0, T^{km(x_0)+s} x_0, \phi(t)) \\
&\leq \mathcal{N}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) \diamond \mathcal{N}(T^{m(x_0)} x_0, T^{km(x_0)+s} x_0, T^{km(x_0)+s} x0, t) \\
&\leq \mathcal{N}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) \diamond \mathcal{N}(x_0, T^{(k-1)m(x_0)+s} x_0, T^{(k-1)m(x_0)+s} x_0, \phi(t)) \\
&\leq \dots \leq \mathcal{N}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) \diamond \dots^{(k)} \diamond \mathcal{N}(x_0, T^{m(x_0)} x_0, T^{m(x_0)} x_0, \phi(t) - t) \\
&\quad \diamond \mathcal{N}(x_0, T^s x_0, T^s x_0, \phi(t)) \\
&< \lambda_k \diamond \dots^{(k+1)} \diamond \lambda_k \\
&< \lambda_{k-1} \diamond \dots^{(k)} \diamond \lambda_{k-1} \\
&< \dots < \lambda_1 \diamond \lambda_1 < \lambda.
\end{aligned}$$

Therefore, for all $t > t_0$,

$$\inf_{x \in O_T(x_0)} \mathcal{M}(x_0, x, x, \phi(t)) \geq 1 - \lambda \text{ and } \sup_{x \in O_T(x_0)} \mathcal{N}(x_0, x, x, \phi(t)) \leq \lambda$$

and hence

$$\sup_{s > 0} \inf_{x \in O_T(x_0)} \mathcal{M}(x_0, x, x, s) = 1 - \lambda \text{ and } \inf_{s > 0} \sup_{x \in O_T(x_0)} \mathcal{N}(x_0, x, x, s) = \lambda.$$

By the arbitrariness of λ, we have

$$\sup_{s > 0} \inf_{x \in O_T(x_0)} \mathcal{M}(x_0, x, x, s) = 1 \text{ and } \inf_{s > 0} \sup_{x \in O_T(x_0)} \mathcal{N}(x_0, x, x, s) = 0.$$

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Next, we prove that the quasi-iterative sequence $\{x_n = T^{m(x_{n-1})}x_{n-1}\}_{n=1}^\infty$ is a Cauchy sequence.

For convenience, put $m_i = m(x_i), i = 0, 1, 2 \ldots$. Then, by (3.4.1), for all $t > 0$, we have

$$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) = \mathcal{M}(T^{m_{n-1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}\cdots+m_{n-1}}x_{n-1},$$
$$T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, t)$$
$$\geq \mathcal{M}(x_{n-1}, T^{m_{n+p-1}+\cdots+m_n}x_{n-1}, T^{m_{n+p-1}+\cdots+m_n}x_{n-1}, \phi(t))$$
$$\geq \mathcal{M}(x_{n-2}, T^{m_{n+p-1}+\cdots+m_n}x_{n-2}, T^{m_{n+p-1}+\cdots+m_n}x_{n-2}, \phi^2(t))$$
$$\geq \cdots \geq \mathcal{M}(x_0, T^{m_{n+p-1}+\cdots+m_n}x_0, T^{m_{n+p-1}+\cdots+m_n}x_0, \phi^n(t))$$
$$\geq \inf_{x\in O_T(x_0)} \mathcal{M}(x_0, x, x, \phi^n(t))$$
$$\geq \sup_{0<s<\phi^n(t)} \inf_{x\in O_T(x_0)} \mathcal{M}(x_0, x, x, s),$$

$$\mathcal{N}(x_n, x_{n+p}, x_{n+p}, t) = \mathcal{N}(T^{m_{n-1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}\cdots+m_{n-1}}x_{n-1},$$
$$T^{m_{n+p-1}+m_{n+p-2}+\cdots+m_{n-1}}x_{n-1}, t)$$
$$\leq \mathcal{N}(x_{n-1}, T^{m_{n+p-1}+\cdots+m_n}x_{n-1}, T^{m_{n+p-1}+\cdots+m_n}x_{n-1}, \phi(t))$$
$$\leq \mathcal{N}(x_{n-2}, T^{m_{n+p-1}+\cdots+m_n}x_{n-2}, T^{m_{n+p-1}+\cdots+m_n}x_{n-2}, \phi^2(t))$$
$$\leq \cdots \leq \mathcal{N}(x_0, T^{m_{n+p-1}+\cdots+m_n}x_0, T^{m_{n+p-1}+\cdots+m_n}x_0, \phi^n(t))$$
$$\leq \sup_{x\in O_T(x_0)} \mathcal{N}(x_0, x, x, \phi^n(t))$$
$$\leq \inf_{0<s<\phi^n(t)} \sup_{x\in O_T(x_0)} \mathcal{N}(x_0, x, x, s)$$

Then, by condition (ϕ_1) and (3.4.3)

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{N}(x_n, x_{n+p}, x_{n+p}, t) = 0$$

for all $t > 0$. This means that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $\lim_{n\to\infty} x_n = x^* \in X$. Now, we prove that x^* is the unique fixed point T^{m^*}, where $m^* = m(x^*)$. By (v) and (xi) in Definition 2.1, and (3.4.2), we have

$$\mathcal{M}(x^*, T^{m^*}x^*, T^{m^*}x^*, t) \geq \mathcal{M}(x^*, T^{m^*}x_n, T^{m^*}x_n, t/2) * \mathcal{M}(T^{m^*}x_n, T^{m^*}x^*, T^{m^*}x^*, t/2)$$
$$\geq \mathcal{M}(x^*, T^{m^*}x_n, T^{m^*}x_n, t/2) * \mathcal{M}(x_n, T^{m^*}x^*, T^{m^*}x^*, \phi(t/2))$$
$$(3.4.6)$$

$$\mathcal{N}(x^*, T^{m^*}x^*, T^{m^*}x^*, t) \leq \mathcal{N}(x^*, T^{m^*}x_n, T^{m^*}x_n, t/2) \diamond \mathcal{N}(T^{m^*}x_n, T^{m^*}x^*, T^{m^*}x^*, t/2)$$
$$\leq \mathcal{N}(x^*, T^{m^*}x_n, T^{m^*}x_n, t/2) \diamond \mathcal{N}(x_n, T^{m^*}x^*, T^{m^*}x^*, \phi(t/2))$$
$$(3.4.7)$$

It is easy to prove that

$$\lim_{n\to\infty} \mathcal{M}(x^*, T^{m^*}x_n, T^{m^*}x_n, u) = 1 \text{ and } \lim_{n\to\infty} \mathcal{N}(x^*, T^{m^*}x_n, T^{m^*}x_n, u) = 0$$

for all $u > 0$. In fact,

$$\mathcal{M}(x^*, T^{m^*}x_n, T^{m^*}x_n, u) \geq \mathcal{M}(x^*, x_n, x_n, u/2) * \mathcal{M}(x^*, T^{m^*}x_n, T^{m^*}x_n, u/2)$$

$$\geq \mathcal{M}(x^*, x_n, x_n, u/2) * \mathcal{M}(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m^*}x_{n-1}, T^{m_{n-1}+m^*}x_n, u/2)$$

$$\geq \mathcal{M}(x^*, x_n, x_n, u/2) * \mathcal{M}(x_{n-1}, T^{m^*}x_{n-1}, T^{m^*}x_{n-1}, \phi(u/2))$$

$$\geq \cdots \geq \mathcal{M}(x^*, x_n, x_n, u/2) * \mathcal{M}(x_0, T^{m^*}x_0, T^{m^*}x_0, \phi^n(u/2)) \to 1.$$

$$\mathcal{N}(x^*, T^{m^*}x_n, T^{m^*}x_n, u) \leq \mathcal{N}(x^*, x_n, x_n, u/2) \diamond \mathcal{N}(x^*, T^{m^*}x_n, T^{m^*}x_n, u/2)$$

$$\leq \mathcal{N}(x^*, x_n, x_n, u/2) \diamond \mathcal{N}(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m^*}x_{n-1}, T^{m_{n-1}+m^*}x_n, u/2)$$

$$\leq \mathcal{N}(x^*, x_n, x_n, u/2) \diamond \mathcal{N}(x_{n-1}, T^{m^*}x_{n-1}, T^{m^*}x_{n-1}, \phi(u/2))$$

$$\leq \cdots \leq \mathcal{N}(x^*, x_n, x_n, u/2) \diamond \mathcal{N}(x_0, T^{m^*}x_0, T^{m^*}x_0, \phi^n(u/2)) \to 0$$

Thus, letting $n \to \infty$ on the right-hand sides of (3.4.6) and (3.4.7), and noting the continuity of $*$ and $\diamond$, we have

$$\mathcal{M}(x^*, T^{m^*}x^*, T^{m^*}x^*, t) = 1 \quad \text{and} \quad \mathcal{N}(x^*, T^{m^*}x^* T^{m^*}x^*, t) = 0$$

for all $t > 0$.

This implies that $T^{m^*}x^* = x^*$, i.e., x^* is a fixed point $T^{m(x^*)}$.

To show uniqueness, assume that $T^{m(x^*)}(y) = y, T^{m(x^*)}(z) = z$ for some $y, z \in X$.

Then,

$$\mathcal{M}(x^*, y, z, t) = \mathcal{M}(T^{m(x^*)}(x^*), T^{m(x^*)}(y), T^{m(x^*)}(z), t) \geq \mathcal{M}(x^*, y, z, \phi(t)),$$

$$\mathcal{N}(x^*, y, z, t) = \mathcal{N}(T^{m(x^*)}(x^*), T^{m(x^*)}(y), T^{m(x^*)}(z), t) \leq \mathcal{N}(x^*, y, z, \phi(t)).$$

On the other hand, as $\mathcal{M}(x^*, y, z, t)$ is non-decreasing and $\mathcal{N}(x^*, y, z, t)$ is non-increasing, we have

$$\mathcal{M}(x^*, y, z, t) \leq \mathcal{M}(x^*, y, z, \phi(t)) \quad \text{and} \quad \mathcal{N}(x^*, y, z, t) \geq \mathcal{N}(x^*, y, z, \phi(t)),$$

respectively.

Thus,

$$\mathcal{M}(x^*, y, z, t) = \mathcal{M}(x^*, y, z, \phi(t)) = \mathcal{M}(x^*, y, z, \phi^n(t)),$$

$$\mathcal{N}(x^*, y, z, t) = \mathcal{N}(x^*, y, z, \phi(t)) = \mathcal{N}(x^*, y, z, \phi^n(t))$$

for all $t > 0$.

Hence, by condition (ϕ_1), and (vii) and (xiii) in Definition 2.1, we have

$$\mathcal{M}(x^*, y, z, t) = 1 \quad \text{and} \quad \mathcal{N}(x^*, y, z, t) = 0,$$

i.e. $x^* = y = z$. Finally, we prove x^* is the unique fixed point of T,

Since $T^{m(x^*)}x^* = x^*$, it follows that $Tx^* = T(T^{m^*}x^*) = T^{m^*}(Tx^*)$.

Hence, $Tx^* = x^*$.

The uniqueness is obvious. This completes the proof. $\qquad\square$

From Theorem 3.4, we can immediately obtain the following consequence.

Corollary 3.5. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete generalized intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:

(i) There exists $x_0 \in X$ such that

$$\lim_{t\to\infty} \mathcal{M}(x_0, T^i x_0, T^i x_0, t) = 1$$

and

$$\lim_{t\to\infty} \mathcal{N}(x_0, T^i x_0, T^i x_0, t) = 0, i = 1, 2, \ldots$$

(ii) There exists a mapping $m : X \to \mathcal{N}$ such that for any $x, y, z \in X$

$$\mathcal{M}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \geq \mathcal{M}(x, y, z, t/k),$$

$$\mathcal{N}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \leq \mathcal{N}(x, y, z, t/k)$$

where $0 < k < 1$.

Then the conclusion of Theorem 3.4 remains true.

Proof. Taking $\phi(t) = t/k$. Obviously, $\phi(t)$ satisfies the conditions (ϕ_1) and (ϕ_2). Therefore, the conclusion directly follows from Theorem 3.4. $\qquad\square$

Corollary 3.6. Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a complete generalized intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping. If there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y, z \in X$,

$$\mathcal{M}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \geq \mathcal{M}(x, y, z, \phi(t)),$$

$$\mathcal{N}(T^{m(x)} x, T^{m(x)} y, T^{m(x)} z, t) \leq \mathcal{N}(x, y, z, \phi(t)),$$

where the function $\phi(t)$ satisfies the conditions (ϕ_1) and (ϕ_2). Then T has a unique fixed point x^* and the iterative sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. By Theorem 3.4, we only need to show that the iterative sequence $\{T^n x\}$ converges to x^*. For $n \in \mathbb{N}$ with $n > m(x^*), n = km(x^*) + s, 0 \leq s < m(x^*)$.

Since

$$
\begin{aligned}
\mathcal{M}(x^*, T^n x, T^n x, t) &= \mathcal{M}(T^{m(x^*)} x^*, T^{km(x^*)+s} x, T^{km(x^*)+s} x, t) \\
&\geq \mathcal{M}(x^*, T^{(k-1)m(x^*)+s} x, T^{(k-1)m(x^*)+s} x, \phi(t)) \\
&\geq \cdots \geq \mathcal{M}(x^*, T^s x, T^s x, \phi^n(t)) \to 1, \\
\mathcal{N}(x^*, T^n x, T^n x, t) &= \mathcal{N}(T^{m(x^*)} x^*, T^{km(x^*)+s} x, T^{km(x^*)+s} x, t) \\
&\leq \mathcal{N}(x^*, T^{(k-1)m(x^*)+s} x, T^{(k-1)m(x^*)+s} x, \phi(t)) \\
&\leq \cdots \leq \mathcal{N}(x^*, T^s x, T^s x, \phi^n(t)) \to 0,
\end{aligned}
$$

so, $\lim_{n\to\infty} \mathcal{M}(x^*, T^n x, T^n x, t) = 1$ and $\lim_{n\to\infty} \mathcal{N}(x^*, T^n x, T^n x, t) = 0$ for all $t > 0$, i.e., $T^n x \to x^*$. $\qquad\square$

Theorem 3.7. *Let $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ be a compact generalized intuitionistic fuzzy metric space. Let $T : X \to X$ be a continuous mapping satisfying*

$$\mathcal{M}(Tx, Ty, Tz, .) > \min\{\mathcal{M}(x, Tx, Tx, .), \mathcal{M}(y, Ty, Ty, .), \mathcal{M}(z, Tz, Tz, .), \mathcal{M}(x, y, z, .)\},$$
(3.7.1)

$$\mathcal{N}(Tx, Ty, Tz, .) < \max\{\mathcal{N}(x, Tx, Tx, .), \mathcal{N}(y, Ty, Ty, .), \mathcal{N}(z, Tz, Tz, .), \mathcal{N}(x, y, z, .)\}$$
(3.7.2)

for all $x \neq y \neq z$. If there exists $x_0 \in X$ such that $\{T^n x_0\}_{n=1}^{\infty}$ has an accumulation point $x^* \in X$ and for all $t > 0, n = 1, 2, \ldots$.

$$\mathcal{M}(T^{n-1}x_0, T^n x_0, T^n x_0, t) \leq \mathcal{M}(T^n x_0, T^{n+1}x_0, T^{n+1}x_0, t),$$
$$\mathcal{N}(T^{n-1}x_0, T^n x_0, T^n x_0, t) \geq \mathcal{N}(T^n x_0, T^{n+1}x_0, T^{n+1}x_0, t)$$

then x^* is the unique fixed point of T and $\lim_{n \to \infty} T^n x_0 = x^*$.

Proof. Assume $T^n x_0 \neq T^{n+1}x_0$ for each $n \in N$. (If not, there is $n_0 \in N$ such that $T^{n_0}x_0 = T^{n_0+1}x_0$. This means that $x^* = T^{n_0}x_0$ is a fixed point of T and $\lim_{n \to \infty} T^n x_0 = x^*$).

Since $\{T^n x_0\}_{n=1}^{\infty}$ has an accumulation point $x^* \in X$, there exists a subsequence $\{T^{n_i}x_0\}$, $\lim_{n \to \infty} T^{n_i}x_0 = x^*$.

$$\{\mathcal{M}(T^n x_0, T^{n+1}x_0, T^{n+1}x_0, t)\} \quad \text{and} \quad \{\mathcal{N}(T^n x_0, T^{n+1}x_0, T^{n+1}x_0, t)\}$$

are non-decreasing and non-increasing, respectively, and also bounded, thus

$$\{\mathcal{M}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t)\}, \{\mathcal{M}(T^{n_i+1}x_0, T^{n_i+2}x_0, T^{n_i+2}x_0, t)\}$$

and

$$\{\mathcal{N}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t)\}, \{\mathcal{N}(T^{n_i+1}x_0, T^{n_i+2}x_0, T^{n_i+2}x_0, t)\}$$

are convergent to a common limit, i.e.,

$$\lim_{i \to \infty} \mathcal{M}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) = \lim_{i \to \infty} \mathcal{M}(T^{n_i+1}x_0, T^{n_i+2}x_0, T^{n_i+2}x_0, t)$$

$$\lim_{i \to \infty} \mathcal{N}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) = \lim_{i \to \infty} \mathcal{N}(T^{n_i+1}x0, T^{n_i+2}x_0, T^{n_i+2}x_0, t)$$

for all $t > 0$.

By the continuity of T, we have

$$\lim_{i \to \infty} T^{n_i+1}x_0 = \lim_{i \to \infty} T(T^{n_i}x_0) = Tx^*.$$

Suppose $x^* \neq Tx^*$. Putting $y = Tx, z = Tx$ in (3.7.1) and (3.7.2), we have

$$\mathcal{M}(x, Tx, Tx, .) < \mathcal{M}(x, T^2x, T^2x, .) \quad \text{and} \quad \mathcal{N}(x, Tx, Tx, .) > \mathcal{N}(x, T^2x, T^2x, .)$$

for every $x \neq Tx$. So, by Lemma 3.3, there exists a continuous point t_0 of $\mathcal{M}(x^*, Tx^*, Tx^*, .)$ and $\mathcal{N}(x^*, Tx^*, Tx^*, .)$ such that

$$\mathcal{M}(x^*, T^2x^*, T^2x^*, t_0) > \mathcal{M}(x^*, Tx^*, Tx^*, t_0) \text{ and}$$

$$\mathcal{N}(x^*, T^2x^*, T^2x^*, t_0) < \mathcal{N}(x^*, Tx^*, Tx^*, t_0).$$

On the other hand, by Lemma 2.5.

$$
\begin{aligned}
\mathcal{M}(x^*, Tx^*, Tx^*, t_0) &= \lim_{i\to\infty} \mathcal{M}(T^{n_i}x_0, T(T^{n_i}x_0), T(T^{n_i}x_0), t_0) \\
&= \lim_{i\to\infty} \mathcal{M}(T^{n_i+1}x_0, T^{n_i+2}x_0, T^{n_i+2}x_0, t_0) \\
&\geq \mathcal{M}(Tx^*, T^2x^*, T^2x^*, t_0) \\
\mathcal{N}(x^*, Tx^*, Tx^*, t_0) &= \lim_{i\to\infty} \mathcal{N}(T^{n_i}x_0, T(T^{n_i}x_0), T(T^{n_i}x_0), t_0) \\
&= \lim_{i\to\infty} \mathcal{N}(T^{n_i+1}x_0, T^{n_i+2}x_0, T^{n_i+2}x_0, t_0) \\
&\leq \mathcal{N}(Tx^*, T^2x^*, T^2x^*, t_0)
\end{aligned}
$$

are contradictions. Therefore, $x^* = Tx^*$, i.e. x^* is a fixed point of T.

Uniqueness follows from (3.7.1) and (3.7.2). Finally, we prove that $\lim_{n\to\infty} T^n x_0 = x^*$. Since $\lim_{i\to\infty} T^{n_i}x_0 = x^*$ and $\lim_{i\to\infty} T^{n_i+1}x_0 = Tx^* = x^*$, by Lemma 2.5,

$$\liminf_{i\to\infty} \mathcal{M}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) \geq \mathcal{M}(x^*, x^*, x^*, t) = 1,$$

$$\limsup_{i\to\infty} \mathcal{N}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) \leq \mathcal{N}(x^*, x^*, x^*, t) = 0,$$

for all $t > 0$. So,

$$\lim_{i\to\infty} \mathcal{M}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) = 1,$$

$$\lim_{i\to\infty} \mathcal{N}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t) = 0,$$

for all $t > 0$. For any $n \in N$ with $n > n_1$, there exists n_i with $n_{i+1} \geq n > n_i$.

By (3.7.1) and (3.7.2), we have

$$
\begin{aligned}
\mathcal{M}(T^n x_0, x^*, x^*, t) &\geq \min\{\mathcal{M}(T^{n-1}x_0, T^n x_0, T^n x_0, t), 1, \mathcal{M}(T^n x_0, x^*, x^*, t)\} \\
&\geq \min\{\mathcal{M}(T^{n-1}x_0, T^n x_0, T^n x_0, t), \mathcal{M}(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0, t), M(T^{n-2}x_0, x^*, x^*, t)\} \\
&\geq \min\{\mathcal{M}(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0, t), \mathcal{M}(T^{n-2}x_0, x^*, x^*, t)\} \\
&\geq \cdots \geq \min\{\mathcal{M}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t), \mathcal{M}(T^{n_i}x_0, x^*, x^*, t)\} \\
\mathcal{N}(T^n x_0, x^*, x^*, t) &\leq \max\{\mathcal{N}(T^{n-1}x_0, T^n x_0, T^n x_0, t), 0, \mathcal{N}(T^n x_0, x^*, x^*, t)\} \\
&\leq \max\{\mathcal{N}(T^{n-1}x_0, T^n x_0, T^n x_0, t), \mathcal{N}(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0, t), \mathcal{N}(T^{n-2}x_0, x^*, x^*, t)\} \\
&\leq \max\{\mathcal{N}(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0, t), \mathcal{N}(T^{n-2}x_0, x^*, x^*, t)\} \\
&\leq \cdots \leq \max\{\mathcal{N}(T^{n_i}x_0, T^{n_i+1}x_0, T^{n_i+1}x_0, t), \mathcal{N}(T^{n_i}x_0, x^*, x^*, t)\}
\end{aligned}
$$

Letting $n \to \infty (n_i \to \infty)$, we have

$$\lim_{n\to\infty} \mathcal{M}(T^n x_0, x^*, x^*, t) \geq 1 \text{ and } \lim_{n\to\infty} \mathcal{N}(T^n x_0, x^*, x^*, t) \leq 0$$

for all $t > 0$.

Hence, $\lim_{n\to\infty} T^n x_0 = x^*$. □

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