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Intuitionistic fuzzy basis of an intuitionistic fuzzy vector space

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Abstract: In the present paper the notion of intuitionistic fuzzy vector space is introduced and a representation theorem is established. The notion of intuitionistic fuzzy basis has been developed.
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1 Introduction

After the introduction of fuzzy sets by Zadeh [21], several generalizations have been made of this fundamental concept for various objectives. The notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1,2] is one among them. There are situations where IFS theory is more appropriate to dealt with [4]. IFS theory is quite interesting and useful in many application areas viz. medical diagnosis [8], decision making [20], career determination [9] etc. Many researchers have been involved in extending various mathematical aspects such as groups, rings, modules, topological spaces, topological groups, topological vector spaces in IFS [3,6,7,10,11,13–16,18]. In 1977, Katsaras introduced the concept of fuzzy vector subspaces [12]. In 2010, a notion of fuzzy bases have been studied in [19]. The notion of intuitionistic fuzzy subspace of a vector space was introduced by many authors [5, 13, 17]. In this paper we introduce a notion of intuitionistic fuzzy vector space (IFVS) and intuitionistic fuzzy basis (IF-basis) of a IFVS which is analogous to the fuzzy basis of [19].

2 Preliminaries

Definition 2.1 ([1]). *Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X defined as an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.*

In this paper, we use the symbols $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Definition 2.2 ([1]). *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a set X. Then*

(1) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$.

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$

(4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

(5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

(6) $\Box\, A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$, $\Diamond\, A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3 ([6]). $0_\sim = (0, 1)$ *and* $1_\sim = (1, 0)$.

Definition 2.4 ([6]). *Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let A be an IFS in X and B be an IFS in Y. Then*
(a) the image of A under f, denoted by $f(A)$, is the IFS in Y defined by $f(A) = (f(\mu_A), f(\nu_A))$, where for each $y \in Y$,

$$f(\mu_A)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu_A(x) & if\ f^{-1}(y) \neq \phi \\ 0 & if\ f^{-1}(y) = \phi \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge\limits_{x \in f^{-1}(y)} \mu_A(x) & if\ f^{-1}(y) \neq \phi \\ 1 & if\ f^{-1}(y) = \phi \end{cases}$$

(b) the pre-image of B under f, denoted by $f^{-1}(B)$, is the IFS in X, defined by $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B))$, where $f^{-1}(\mu_B) = \mu_B \circ f$.

Corollary 2.5 ([6]). *Let $A, \{A_i\}_{i \in J}$ be IFS in X, $B, \{B_j\}_{j \in K}$ be IFS in Y and $f : X \to Y$ be a mapping. Then*

(1) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$.

(2) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f(B_2)$.

(3) $A \subseteq f^{-1}(f(A))$. If f is injective, $A = f^{-1}(f(A))$.

(4) $f(f^{-1}(B)) \subseteq B$. *If f is surjective,* $f(f^{-1}(B)) = B$.

(5) $f(\cup A_i) = \cup f(A_i)$.

(6) $f(\cap A_i) \subseteq \cap f(A_i)$. *If f is injective,* $f(\cap A_i) = \cap f(A_i)$.

(7) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$.

(8) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$.

(9) $f(1_\sim) = 1_\sim$, *if f is surjective and* $f(0_\sim) = 0_\sim$.

(10) $f^{-1}(1_\sim) = 1_\sim$ *and* $f^{-1}(1_\sim) = 1_\sim$.

(11) $[f(A)]^c \subseteq f(A^c)$, *if f is surjective.*

(12) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Definition 2.6 ([10]). *Let A be an IFS in a set X. Then for $\lambda, \xi \in [0,1]$ with $\lambda + \xi \leq 1$, the set $A^{[\lambda,\xi]} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \xi\} = \{x \in X : A(x) \geq (\lambda, \xi)\}$ is called (λ, ξ)-level subset of A.*

Proposition 2.7 ([10]). *Let A be an IFS in a set X and $(\lambda_1, \xi_1), (\lambda_2, \xi_2) \in Im(A)$. If $\lambda_1 \leq \lambda_2$ and $\xi_1 \geq \xi_2$, then $A^{[\lambda_1,\xi_1]} \supseteq A^{[\lambda_2,\xi_2]}$.*

Definition 2.8 ([13]). *Let X be a vector space over the field K, the field of real and complex numbers, $\alpha \in K$, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of X.Then*

(1) *the sum of A and B is defined to be the intuitionistic fuzzy set $A + B = (\mu_A + \mu_B, \nu_A + \nu_B)$ of X given by*

$$\mu_{A+B}(x) = \begin{cases} \sup\limits_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} & if\ x = a + b \\ 0 & otherwise, \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \inf\limits_{x=a+b} \{\nu_A(a) \vee \nu_B(b)\} & if\ x = a + b \\ 1 & otherwise. \end{cases}$$

(2) *αA is defined to be the IFS $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ of X, where*

$$\mu_{\alpha A}(x) = \begin{cases} \mu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \sup\limits_{y \in X} \mu_A(y) & if\ \alpha = 0, x = \theta \\ 0 & if\ \alpha = 0, x \neq \theta, \end{cases}$$

$$\nu_{\alpha A}(x) = \begin{cases} \nu_A(\alpha^{-1}x) & if\ \alpha \neq 0 \\ \inf\limits_{y \in X} \mu_A(y) & if\ \alpha = 0, x = \theta\ . \\ 1 & if\ \alpha = 0, x \neq \theta. \end{cases}$$

Remark 2.9. *Let X be a vector space over the field K, the field of real and complex numbers, $A = (\mu_A, \nu_A)$ an intuitionistic fuzzy set of X . Then for all scalars $\alpha \in K$ and for all $x \in X$, we have $\mu_{\alpha A}(\alpha x) \geq \mu_A(x)$ and $\nu_{\alpha A}(\alpha x) \leq \nu_A(x)$.*

Proposition 2.10. *Let $A, A_1, \ldots, A_n$ be intuitionistic fuzzy sets in a vector space X and $\lambda_1, \ldots, \lambda_n$ be scalars. Then the following assertions are equivalent:*

(1) $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n \subseteq A$.

(2) For all $x_1, x_2, \ldots, x_n$ in X, we have $\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \geq \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}$ *and* $\nu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \max\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots, \nu_{A_n}(x_n)\}$.

Proof. $(1) \Rightarrow (2)$:
$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \geq \mu_{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n}(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$
$\geq \min\{\mu_{\lambda_1 A_1}(\lambda_1 x_1), \ldots, \mu_{\lambda_n A_n}(\lambda_n x_n)\}$
$\geq \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\}$.
$\nu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \nu_{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n}(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$
$\leq \max\{\nu_{\lambda_1 A_1}(\lambda_1 x_1), \ldots, \nu_{\lambda_n A_n}(\lambda_n x_n)\}$
$\leq \max\{\nu_{A_1}(x_1), \ldots, \nu_{A_n}(x_n)\}$.
$(2) \Rightarrow (1)$:
By rearranging the order if necessary, we may assume that $\lambda_i \neq 0$ for $i = 1, 2, \ldots, k$, and $\lambda_i = 0$ for $k \leq i \leq n$. Let $x_1, x_2, \ldots, x_k$ be elements of X. For all $y_1, y_2, \ldots, y_{n-k}$ in X we have
$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) \geq \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_k}(x_k), \mu_{A_{k+1}}(y_1), \ldots, \mu_{A_n}(y_{n-k})\}$.
Since $\mu_{0A_j}(\theta) = \sup_{y \in X} \mu_{A_j}(y)$, we get
$\mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) \geq \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_k}(x_k), \mu_{0A_{k+1}}(\theta), \ldots, \mu_{0A_n}(\theta)\}$.
Now
$\mu_{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n}(z) = \sup_{x_1 + x_2 + \cdots + x_n = z} [\min\{\mu_{\lambda_1 A_1}(x_1), \ldots, \mu_{\lambda_n A_n}(x_n)\}]$
$= \sup_{x_1 + x_2 + \cdots + x_n = z} [\min\{\mu_{\lambda_1 A_1}(x_1), \ldots, \mu_{\lambda_k A_k}(x_k), \mu_{0A_{k+1}}(x_{k+1}), \ldots, \mu_{0A_n}(x_n)\}]$
$= \sup_{x_1 + x_2 + \cdots + x_k = z} [\min\{\mu_{A_1}(\lambda_1^{-1} x_1), \ldots, \mu_{A_k}(\lambda_k^{-1} x_k), \mu_{0A_{k+1}}(\theta), \ldots, \mu_{0A_n}(\theta)\}]$[Since $\mu_{0A_i}(x_i) = 0$, if $x_i \neq \theta$, $i = k+1, \ldots, n$]
$\leq \sup_{x_1 + x_2 + \cdots + x_k = z} \mu_A(\lambda_1 \lambda_1^{-1} x_1 + \cdots + \lambda_k \lambda_k^{-1} x_k) = \mu_A(z)$.
Similarly, it can be proved that $\nu_{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n}(z) \geq \nu_A(z),\ z \in X$.
Hence proved. □

Proposition 2.11. *Let A, B be two intuitionistic fuzzy sets in a vector space X. Then*

(1) $A + 0B \subseteq A$.

(2) $A + 0B = A$ *iff* $\sup_{x \in X} \mu_A(x) \leq \sup_{x \in X} \mu_B(x)$ *and* $\inf_{x \in X} \nu_A(x) \geq \inf_{x \in X} \nu_B(x)$.

Proof. (1) $\mu_A(x + 0y) = \mu_A(x) \geq \min\{\mu_A(x), \mu_B(y)\}$ and $\nu_A(x + 0y) = \nu_A(x) \leq \max\{\nu_A(x), \nu_B(y)\}$. Hence (1) follows from $Proposition$ 2.9.

(2) Suppose that $\sup_{x \in X} \mu_A(x) \leq \sup_{x \in X} \mu_B(x) = \mu_{0B}(\theta)$ and $\inf_{x \in X} \nu_A(x) \geq \inf_{x \in X} \nu_B(x) = \nu_{0B}(\theta)$
Then $\mu_{A+0B}(z) = \sup_{x+y=z} [\min\{\mu_A(x), \mu_{0B}(y)\}] = \min\{\mu_A(z), \mu_{0B}(\theta)\} = \mu_A(z)$ and

$\nu_{A+0B}(z) = \inf_{x+y=z}[max\{\nu_A(x), \nu_{0B}(y)\}] = max\{\nu_A(z), \nu_{0B}(\theta)\} = \nu_A(z)$
On the other hand, if $\mu_A(z) > sup\, \mu_B(x) = \mu_{0B}(\theta)$ for some z, then

$$\mu_{A+0B}(z) = \min\{\mu_A(z), \mu_B(0)\} < \mu_A(z),$$

and hence $A + 0B \neq A$. □

Proposition 2.12. *Let X and Y be two vector spaces and $f : X \to Y$ be a linear onto map. Then for all IFS A, B of X and for all scalars k,*
(1) $f(A+B) = f(A) + f(B)$
(2) $f(kA) = kf(A)$.

Proof. (1) Let $y \in Y$ and $\varepsilon > 0$ be arbitrary. Let $(\alpha, \alpha') = f(A+B)(y)$ and $(\beta, \beta') = (f(A)+f(B))(y)$. Then: $\alpha = \mu_{f(A+B)(y)} = \bigvee_{z \in f^{-1}(y)} \mu_{A+B}(z)$, $\alpha' = \nu_{f(A+B)(y)} = \bigwedge_{z \in f^{-1}(y)} \nu_{A+B}(z)$ and $\beta = \mu_{(f(A)+f(B))(y)} = \bigvee_{y=z+z'}[\mu_{f(A)}(z) \wedge \mu_{f(B)}(z')]$, $\beta' = \nu_{(f(A)+f(B))(y)} = \bigwedge_{y=z+z'}[\nu_{f(A)}(z) \vee \nu_{f(A)}(z')]$.
Thus $\alpha - \varepsilon < \bigvee_{z \in f^{-1}(y)} \mu_{A+B}(z)$ and $\alpha' + \epsilon > \bigwedge_{z \in f^{-1}(y)} \nu_{A+B}(z)$. So, there exists $z_0, z_0' \in X$ such that $f(z_0) = y$ and $f(z_0') = y$ such that $\alpha - \varepsilon < \mu_{A+B}(z_0)$ and $\alpha' + \epsilon > \nu_{A+B}(z_0')$. By the definition of sum,
$\alpha - \varepsilon < \bigvee_{z_0=a+b}[\mu_A(a) \wedge \mu_B(b)]$ and $\alpha' + \epsilon > \bigwedge_{z_0'=a'+b'}[\nu_A(a') \vee \nu_B(b')]$. Then there exist $a_0, b_0 \in X$ with $z_0 = a_0 + b_0$ such that $\alpha - \varepsilon < \mu_A(a_0) \wedge \mu_B(b_0)$ and there exist $a_0', b_0' \in X$ with $z_0' = a_0' + b_0'$ such that $\alpha' + \epsilon > \nu_A(a_0') \vee \nu_B(b_0')$. On the other hand,
$\beta \geq \mu_{f(A)}(f(a_0)) \wedge \mu_{f(B)}(f(b_0))$
$= f(\mu_A)(f(a_0)) \wedge f(\mu_B)(f(b_0))$
$= f^{-1}(f(\mu_A))(a_0) \wedge f^{-1}(f(\mu_B))(b_0)$
$\geq \mu_A(a_0) \wedge \mu_B(b_0)$.
Similarly we have, $\beta' \leq \nu_A(a_0') \vee \nu_B(b_0')$. So, $\beta > \alpha - \varepsilon$ and $\beta' < \alpha' + \varepsilon$. Since ε is arbitrary, $\beta \geq \alpha$ and $\beta' \leq \alpha'$. Hence $(f(A) + f(B))(y) \geq f(A+B)(y)$, for each $y \in Y$. (*)
Now we will show that $\beta \leq \alpha$ and $\beta' \geq \alpha'$. Clearly,
$\beta - \varepsilon < \bigvee_{y=z+z'}[\mu_{f(A)}(z) \wedge \mu_{f(B)}(z')]$ and $\beta' + \epsilon > \bigwedge_{y=z+z'}[\nu_{f(A)}(z) \vee \nu_{f(A)}(z')]$.
Then there exist $z_0, z_0' \in Y$ with $y = z_0 + z_0'$such that $\beta - \varepsilon < \mu_{f(A)}(z_0) = \bigvee_{x \in f^{-1}(z_0)} \mu_A(x)$ and $\beta - \varepsilon < \mu_{f(B)}(z_0') = \bigvee_{x \in f^{-1}(z_0')} \mu_B(x)$
and there exist $z_1, z_1' \in Y$ with $y = z_1 + z_1'$such that $\beta' + \varepsilon > \nu_{f(A)}(z_1) = \bigwedge_{x \in f^{-1}(z_1)} \nu_A(x)$ and $\beta' + \varepsilon > \nu_{f(B)}(z_1') = \bigwedge_{x \in f^{-1}(z_1')} \nu_B(x)$.
Thus there exist $x_0, x_0' \in X$ with $f(x_0) = z_0$ and $f(x_0') = z_0'$ such that $\beta - \varepsilon < \mu_A(x_0)$, $\beta - \varepsilon < \mu_B(x_0')$
and there exist $x_1, x_1' \in X$ with $f(x_1) = z_1$ and $f(x_1') = z_1'$ such that $\beta' + \varepsilon > \nu_A(x_1)$, $\beta' + \varepsilon > \nu_B(x_1')$.
So, $\beta - \varepsilon < \mu_A(x_0) \wedge \mu_B(x_0') \leq \mu_{A+B}(x_0 + x_0') \leq \bigvee_{x \in f^{-1}(y)} \mu_{A+B}(x) = \mu_{f(A+B)}(y)$
and $\beta' + \varepsilon > \nu_A(x_1) \vee \nu_B(x_1') \geq \nu_{A+B}(x_1 + x_1') \geq \bigwedge_{x \in f^{-1}(y)} \nu_{A+B}(x) = \nu_{f(A+B)}(y)$.

Hence $\beta - \varepsilon < \alpha$ and $\beta' + \varepsilon > \alpha'$. Since ε is arbitrary, $\beta \leq \alpha$ and $\beta' \geq \alpha'$. Hence $(f(A) + f(B))(y) \leq f(A + B)(y)$, for each $y \in Y$. (**)
Therefore, by (*) and (**), $f(A) + f(B) = f(A + B)$.

(2) Let $y \in Y$, $(\alpha, \alpha') = (kf(A))(y)$ and $(\beta, \beta') = (f(kA))(y)$. If $k \neq 0$, $\alpha = \mu_{f(A)}(\lambda^{-1}y) = \bigvee_{f(x)=\lambda^{-1}y} \mu_A(x) = \bigvee_{f(\lambda x)=y} \mu_{\lambda A}(\lambda x)$
$= \bigvee_{f(z)=y} \mu_{\lambda A}(z) = \beta$.
Next assume that $k = 0$. If $y \neq \theta$, then $\alpha = 0$. Also $\beta = \bigvee_{f(x)=y} \mu_{0A}(x) = 0$ since, when $f(x) = y \neq \theta$, $x \neq \theta$. For $y = \theta$, we have
$\alpha = \bigvee_{y \in Y} \mu_{f(A)}(y) = \bigvee_{x \in X} \mu_A(x)$;
$\beta = \bigvee_{f(x)=\theta} \mu_{0A}(x) = \mu_{0A}(\theta) = \bigvee_{x \in X} \mu_A(x)$.
Similarly, it can be proved that $\alpha' = \beta'$.
This completes the proof. □

3 Intuitionistic fuzzy vector space

Definition 3.1. *An IFS $V = (\mu_V, \nu_V)$ of a vector space X over the field K is said to be intuitionistic fuzzy vector space over X if*

(i) $V + V \subseteq V$

(ii) $\alpha V \subseteq V$, for every scalar α.

We denote the set of all intuitionistic fuzzy vector spaces over a vector space X by IFVS(X).

Remark 3.2. *Let X be a vector space.*

(1) If μ_V is a fuzzy subspace of X, then $V = (\mu_V, \mu_V^c) \in$ IFVS(X).

(2) If $V \in$ IFVS(X), then μ_V and ν_V^c are fuzzy vector subspace of X.

(3) If $V \in$ IFVS(X), then $\Box\, V, \Diamond\, V \in$ IFVS(X).

Lemma 3.3. *Let V be an intuitionistic fuzzy set in a vector space X. Then, the following are equivalent:*

(1) V is an intuitionistic fuzzy vector space over X.

(2) For all scalars α, β, we have $\alpha V + \beta V \subseteq V$.

(3) For all scalars α, β and for all $x, y \in X$, we have
$\mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y)\}$ and $\nu_V(\alpha x + \beta y) \leq \nu_V(x) \vee \nu_V(y)$.

Proof. Clearly, $(1) \Rightarrow (2)$. Also (2) and (3) are equivalent by $Proposition$ 2.10.
$(2) \Rightarrow (1) : V + V = 1V + 1V \subseteq V, \alpha V = \alpha V + 0V \subseteq V$. □

Remark 3.4. *Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [17] and [5].*

Proposition 3.5. *Let X and Y be vector spaces over K and let f be a linear map from X onto Y. If V is an intuitionistic fuzzy vector space over X, then $f(V)$ is an intuitionistic fuzzy vector space over Y. Similarly, if W is an intuitionistic fuzzy vector space over Y, then $f^{-1}(W)$ is an intuitionistic fuzzy vector space over X.*

Proof. For k, m scalars, we have from Proposition 2.12, $kf(V) + mf(V) = f(kV + mV) \subseteq f(V)$, which shows that $f(V)$ is an intuitionistic fuzzy vector space over Y. Also,
$\mu_{f^{-1}(W)}(kx + my) = \mu_W(f(kx + my)) = \mu_W(kf(x) + mf(y)) \geq \mu_W(f(x)) \wedge \mu_W(f(y)) = \mu_{f^{-1}(W)}(x) \wedge \mu_{f^{-1}(W)}(y)$ and
$\nu_{f^{-1}(W)}(kx + my) = \nu_W(f(kx + my)) = \nu_W(kf(x) + mf(y)) \leq \nu_W(f(x)) \vee \nu_W(f(y)) = \nu_{f^{-1}(W)}(x) \vee \nu_{f^{-1}(W)}(y)$.
Hence $f^{-1}(W)$ is an intuitionistic fuzzy vector space by $Lemma$ 3.3. □

Proposition 3.6 ([5])**.** *If $V, W \in$ IFVS(X), then $V + W \in$ IFVS(X).*

Proposition 3.7. *If $V \in$ IFVS(X) $\alpha \in K$, then $\alpha V \in$ IFVS(X).*

Proof. We have for $x, y \in X$ and $k, m \in K$, $\mu_V(kx+my) \geq \mu_V(x) \wedge \mu_V(y)$ and $\nu_V(kx+my) \leq \nu_V(x) \vee \nu_V(y)$. Let α be any scalar so that $\alpha \neq 0$, then $\mu_{\alpha V}(kx+my) = \mu_V(\alpha^{-1}kx+\alpha^{-1}my) \geq \mu_V(\alpha^{-1}x) \wedge \mu_V(\alpha^{-1}y)$[by Lemma 3.3] $= \mu_{\alpha V}(x) \wedge \mu_{\alpha V}(y)$ and similarly, $\nu_{\alpha V}(kx + my) \leq \nu_{\alpha V}(x) \vee \nu_{\alpha V}(y)$.

On the other hand, if $\alpha = 0$, then $\mu_{0V}(kx+my) = \begin{cases} 0 & if\ kx + my \neq \theta \\ \sup\limits_{x \in X} \mu_V(x) & if\ kx + my = \theta \end{cases}$ and $\nu_{0V}(kx+my) = \begin{cases} 1 & if\ kx + my \neq \theta \\ \inf\limits_{x \in X} \nu_V(x) & if\ kx + my = \theta \end{cases}$.
If $kx+my = \theta$, $\mu_{0V}(kx+my) = \sup\limits_{x \in X} \mu_V(x) \geq \mu_{0V}(x) \wedge \mu_{0V}(y)$ and $\nu_{0V}(kx+my) = \inf\limits_{x \in X} \nu_V(x) \leq \nu_{0V}(x) \vee \nu_{0V}(y)$.

If $kx + my \neq \theta$, we have $\mu_{0V}(kx + my) = 0$ and $\nu_{0V}(kx + my) = 1$. we must show that $\mu_{0V}(x) \wedge \mu_{0V}(y) = 0$ and $\nu_{0V}(x) \vee \nu_{0V}(y) = 1$. Assume that $\mu_{0V}(x) \wedge \mu_{0V}(y) \neq 0$, then $\mu_{0V}(x) > 0$ and $\mu_{0V}(y) > 0$. So, $y = x = \theta$, a contradiction. Similarly, it can be shown that $\nu_{0V}(x) \vee \nu_{0V}(y) = 1$. □

Proposition 3.8. *[5] If $\{V_i\}_{i \in I} \in$ IFVS(X), then $\bigcap\limits_{i \in I} V_i \in$ IFVS(X).*

Proposition 3.9. *Let $V \in$ IFVS(X). Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x), \forall x \in X$.*

Proof. $0V \subseteq V$. Thus by $Proposition$ 2.10, $\mu_V(\theta) = \mu_V(0.x) \geq \mu_V(x)$ and $\nu_V(\theta) = \nu_V(0.x) \leq \nu_V(x)$, for all $x \in X$. □

Proposition 3.10. *Let $V \in$ IFVS(X). Then for each $(\lambda, \xi) \in [0, 1] \times [0, 1]$ with $\lambda + \xi \leq 1,\ \lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$, $V^{[\lambda,\xi]}$ is a subspace of the vector space X,* *Proof.* Clearly, $V^{[\lambda,\xi]} \neq \phi$. Let $x, y \in V^{[\lambda,\xi]}$ and $k, m \in K$. Then $\mu_V(x), \mu_V(y) \geq \lambda$ and $\nu_V(x), \nu_V(y) \leq \xi$. Since $V \in$ IFVS(X), $\mu_V(kx+my) \geq \mu_V(x) \wedge \mu_V(y) \geq \lambda$ and $\nu_V(kx+my) \leq \nu_V(x) \vee \nu_V(y) \leq \xi$. So, $kx + my \in V^{[\lambda,\xi]}$. Hence $V^{[\lambda,\xi]}$ is a subspace of the vector space X. □

Proposition 3.11. *Let V be an IFS in a vector space X such that $V^{[\lambda,\xi]}$ is a subspace of X for each $(\lambda, \xi) \in [0,1] \times [0,1]$ with $\lambda + \xi \leq 1$, $\lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$. Then $V \in$ IFVS(X).*

Proof. Let $x, y \in X$, $k, m \in K$ and $\mu_V(x) = t_1, \mu_V(y) = t_2$ and $\nu_V(x) = s_1, \nu_V(y) = s_2$. Let $t = t_1 \wedge t_2$ and $s = s_1 \vee s_2$. Then $x, y \in V^{[t,s]}$. Also, if $s = s_1$, $t + s \leq t_1 + s_1 \leq 1$, or if $s = s_2$, then $t + s \leq t_2 + s_2 \leq 1$. Since, $V^{[t,s]}$ is a subspace of X , $kx + my \in V^{[t,s]}$. Then $\mu_V(kx + my) \geq t = \mu_V(x) \wedge \mu_V(y)$ and $\nu_V(kx + my) \leq s = \nu_V(x) \vee \nu_V(y)$. Hence $V \in$ IFVS(X). □

Proposition 3.12. *If $V \in$ IFVS(X), then $V^* = \{x \in X : \mu_V(x) = \mu_V(\theta)$ and $\nu_V(x) = \nu_V(\theta)\}$ is a vector subspace of X.*

Proof. Let $x, y \in V^*$ and $k, m \in K$. Then $\mu_V(x) = \mu_V(\theta)$, $\nu_V(x) = \nu_V(\theta)$ and $\mu_V(y) = \mu_V(\theta)$, $\nu_V(y) = \nu_V(\theta)$. Thus $\mu_V(kx + my) \geq \mu_V(x) \wedge \mu_V(y) = \mu_V(\theta)$ and $\nu_V(kx + my) \leq \nu_V(x) \vee \nu_V(y) = \nu_V(\theta)$. On the other hand, by $Proposition$ 3.9, $\mu_V(kx + my) \leq \mu_V(\theta)$ and $\nu_V(kx+my) \geq \nu_V(\theta)$. So, $\mu_V(kx+my) = \mu_V(\theta)$ $\nu_V(kx+my) = \nu_V(\theta)$. Thus $kx+my \in V^*$. Hence V^* is a subspace of X. □

Proposition 3.13. *Let $s, t \in \mathbb{R}$ and A, A_1 and A_2 be IFS in a vector space X. Then*

(1) $s.(t.A) = t.(s.A) = (st).A$ *and*

(2) $A_1 \leq A_2 \Rightarrow t.A_1 \leq t.A_2$.

Proof. (1) If $s, t \neq 0$:
$s.(t.\nu_A)(x) = (t.\nu_A)(\frac{x}{s})$
$= (\nu_A(\frac{x}{st}))$
$= (s.\nu_A)(\frac{x}{t})$
$= t.(s.\nu_A)(x)$
Also, $(st).\nu_A(x) = \nu_A(\frac{x}{st})$. Similarly, $(st).\mu_A(x) = \mu_A(\frac{x}{st})$.
If $s = 0$ and $t \neq 0$:

$$0.(t.\nu_A)(x) = \begin{cases} \inf\limits_{x \in X} & (t.\nu_A)(x) \; if \; x = \theta \\ 1 & if \; x \neq \theta \end{cases}$$

$$= \begin{cases} \inf\limits_{x \in X} & \nu_A(x) \; if \; x = \theta \\ 1 & if \; x \neq \theta. \end{cases}$$

As $\inf\limits_{x \in X} \nu_A(x) = \inf\limits_{x \in X} \nu_A(\frac{x}{t})$(replace x by tx).
$t.(0.\nu_A)(x) = (0.\nu_A)(\frac{x}{t})$

$$= \begin{cases} \inf\limits_{x \in X} & \nu_A(x) \; if \; \frac{x}{t} = \theta \\ 1 & if \; \frac{x}{t} \neq \theta \end{cases}$$

$= \begin{cases} \inf\limits_{x \in X} & \nu_A(x)\ if\ x = \theta \\ 1 & if\ x \neq \theta. \end{cases}$
$(0t).\nu_A(x) = 0.\nu_A(x)$
$= \begin{cases} \inf\limits_{x \in X} & C_A(x)\ if\ x = \theta \\ 1 & if\ x \neq \theta. \end{cases}$
Similar result holds for μ_A. Obviously, the case where $t = 0$ and $s \neq 0$ is same as the preceding case.
If $s = t = 0$:
$0.(0.\nu_A)(x) = \begin{cases} \inf\limits_{x \in X} & (0.\nu_A)(x)\ if\ x = \theta \\ 1 & if\ x \neq \theta \end{cases}$
$= \begin{cases} \inf\limits_{x \in X} & \nu_A(x)\ if\ x = \theta \\ 1 & if\ x \neq \theta \end{cases}$
$= 0.\nu_A(x)$.
Analogous result holds for μ_A. Hence (1) is proved.
(2) Choose $x \in X$. We have that $\mu_{A_1}(x) \leq \mu_{A_2}(x)$ and If $t \neq 0$, then
$t.\mu_{A_1}(x) = \mu_{A_1}(\frac{x}{t})$
$\leq \mu_{A_2}(\frac{x}{t})$
$= t.\mu_{A_2}(x)$.
If $t = 0$ and $x = \theta$, then $0.\mu_{A_1}(\theta) = \sup\limits_{x \in X} \mu_{A_1}(x)$ and $0.\mu_{A_2}(\theta) = \sup\limits_{x \in X} \mu_{A_2}(x)$. Since we have $\sup\limits_{x \in X} \mu_{A_1}(x) \leq \sup\limits_{x \in X} \mu_{A_2}(x)$, so $0.\mu_{A_1}(\theta) \leq 0.\mu_{A_2}(\theta)$. If $t = 0$ and $x \neq \theta$, then $0.\mu_{A_1}(x) = 0 = 0.\mu_{A_2}(x)$. Similarly, it can be proved that $\nu_{A_1}(x) \geq \nu_{A_2}(x) \Rightarrow \nu_{tA_1}(x) \geq \nu_{tA_2}(x)$, for all $x \in X$. Hence proved. □

Proposition 3.14. *Let $V \in$ IFVS(X) Then $x \in X,\ a \neq 0 \Rightarrow \mu_V(ax) = \mu_V(x)$ and $\nu_V(ax) = \nu_V(x)$.*

Proof. $x \in X,\ a \neq 0$
$\Rightarrow \mu_V(ax) = \mu_V(ax + 0x) \geq \mu_V(x) \wedge \mu_V(x) = \mu_V(x)$ and $\nu_V(ax) = \nu_V(ax + 0x) \leq \nu_V(x) \wedge \nu_V(x) = \nu_V(x)$.
Now, replace x by ax and a by $\frac{1}{a}$, to get $\mu_V(x) \geq \mu_V(ax)$ and $\nu_V(x) \leq \nu_V(ax)$.
Therefore $\mu_V(ax) = \mu_V(x)$ and $\nu_V(ax) = \nu_V(x)$. □

Remark 3.15. *For $V \in$ IFVS(X) we assume that $\mu_V(x) \geq \mu_V(y)$ will always imply $\nu_V(x) \leq \nu_V(y)$, $x, y \in X$. In the following example we see that for an intuitionistic fuzzy set V over X with $\mu_V(x) \geq \mu_V(y)$ and $\nu_V(x) \geq \nu_V(y)$, $x, y \in X$, it may happen that $V \notin$ IFVS(X).*

Example 3.16. *Let $X = \mathbb{R}^2$. We define an intuitionistic fuzzy set $V = (\mu_V, \nu_V)$, where $\mu_V : X \to [0, 1]$ and $\nu_V : X \to [0, 1]$ are given by:*

$$\mu_V(x) = \begin{cases} 1,\ if\ x = (0, 0) \\ .5,\ if\ x = (0, a), a \neq 0 \\ .3,\ otherwise. \end{cases}$$

and $\nu_V(x) = \begin{cases} 0, \ if\ x = (0,0) \\ .4, \ if\ x = (0,a), a \neq 0 \\ .2, \ otherwise. \end{cases}$.

Then we see that V is not an intuitionistic fuzzy vector space as $\nu_V((0,2)) = \nu_V((-1,1) + (1,1)) > \nu_V((-1,1)) \wedge \nu_V((1,1))$.

Definition 3.17. *For any $(a,b),(c,d) \in [0,1] \times [0,1]$ with $a + b \leq 1,\ c + d \leq 1$, we say that:*

(1) $(a,b) \geq (c,d)$ if $a \geq b$ and $c \leq d$.

(2) $(a,b) \leq (c,d)$ if $a \leq b$ and $c \geq d$.

(3) $(a,b) > (c,d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.

(4) $(a,b) < (c,d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.

(5) $(a,b) = (c,d)$ if $a = b$ and $c = d$.

Proposition 3.18. *Let $V \in$ IFVS(X) with $dim\ X = m$. Then $Im(V)$ contains at most $m+1$ points of $[0,1] \times [0,1]$.*

Proof. Let V be an intuitionistic fuzzy vector space in X. Then we show that $Im(V)$ can attain at most m different values on points different from θ. Indeed suppose that we can find $x_0, x_1, \dots, x_m \in X \setminus \{\theta\}$ such that $(\mu_V(x_0), \nu_V(x_0)) < (\mu_V(x_1), \nu_V(x_1)) < \cdots < (\mu_V(x_m), \nu_V(x_m))$. Then $x_0 \notin vct\{x_1, x_2, \dots, x_m\}$, where $vct\{x_1, x_2, \dots, x_m\}$ denote the vector space generated by $\{x_1, x_2, ..., x_m\}$. Otherwise we could find $a_1, a_2, ..., a_m \in K$ such that $x_0 = \sum_{i=1}^{m} a_i x_i$ and then since $V \in$ IFVS(X), it follows from $Lemma$ 3.3(3) that $\mu_V(x_0) = \mu_V(\sum_{i=1}^{m} a_i x_i) \geq \min\{\mu_V(x_1), \mu_V(x_2), \dots, \mu_V(x_m)\} = \mu_V(x_1)$ and $\nu_V(x_0) = \nu_V(\sum_{i=1}^{m} a_i x_i) \leq \max\{\nu_V(x_1), \nu_V(x_2), \dots, \nu_V(x_m)\} = \nu_V(x_1)$, which is impossible. Analogously one can show that $x_1 \notin vct\{x_2, \dots, x_m\}, \dots, x_{m-1} \notin vct\{x_m\}$. Since all $x_i \neq \theta$, we thus have
$dim\ vct\ \{x_0, x_1, \dots, x_m\} = 1 + dim\ vct\ \{x_1, x_2, \dots, x_m\} = 2 + dim\ vct\ \{x_2, x_3, \dots, x_m\} = \cdots = m + dim\ vct\ \{x_m\} = m + 1$.
This however is impossible since $dim\ X = m$.
Consequently the range of V is a subset of $[0,1] \times [0,1]$ with at most $m+1$ points. □

Definition 3.19. *Let $V = (\mu_V, \nu_V) \in$ IFVS(X). Then for any $\lambda \in \mu_V(X), \xi \in \nu_V(X)$ we define $\mu_V^{[\lambda]} = \{x \in X : \mu_V(x) \geq \lambda\}$ and $\nu_V^{[\xi]} = \{x \in X : \nu_V(x) \leq \xi\}$, $[\lambda 1_{\mu_V^{[\lambda]}}](x) = \begin{cases} \lambda, & if\ x \in \mu_V^{[\lambda]} \\ 0, & otherwise \end{cases}$,*

$[\xi 1_{\nu_V^{[\xi]}}](x) = \begin{cases} \xi, & if\ x \in \nu_V^{[\xi]} \\ 1, & otherwise \end{cases}$.

Theorem 3.20. *(Representation Theorem) Let $V \in$ IFVS(X) with $dim\ X = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \dots (\lambda_k, \xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \cdots > (\lambda_k, \xi_k) \geq (0,1)$. Then there exist nested collections of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0, \xi_0]} \subsetneq V^{[\lambda_1, \xi_1]} \subsetneq \cdots \subsetneq$*

$V^{[\lambda_k,\xi_k]} = X$ *such that* $\mu_V = \lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee \cdots \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}$ *and* $\nu_V = \xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \cdots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}$. *Also,*

(1) If $(\zeta,\rho),(\eta,\sigma) \in (\lambda_{i+1},\lambda_i] \times [\xi_i,\xi_{i+1})$ *with* $\zeta+\rho \leq 1, \eta+\sigma \leq 1$, *then* $V^{[\zeta,\rho]} = V^{[\eta,\sigma]} = V^{[\lambda_i,\xi_i]}$.

(2) If $(\zeta,\rho) \in (\lambda_{i+1},\lambda_i] \times [\xi_i,\xi_{i+1}), (\eta,\sigma) \in (\lambda_i,\lambda_{i-1}] \times [\xi_{i-1},\xi_i)$ *with* $\zeta+\rho \leq 1, \eta+\sigma \leq 1$, *then* $V^{[\zeta,\rho]} \supsetneqq V^{[\eta,\sigma]}$.

Proof. From *Proposition* 3.10, $V^{[\lambda_i,\xi_i]}$ are subspaces of X, for $i = 0,1,\ldots,k$. As $(\lambda_i,\xi_i) > (\lambda_{i+1},\xi_{i+1})$ for $i = 0,1,..,k-1$, we have nested collections of subspaces of X as $\{\theta\} \subseteq V^{[\lambda_0,\xi_0]} \subsetneqq V^{[\lambda_1,\xi_1]} \subsetneqq \cdots \subsetneqq V^{[\lambda_k,\xi_k]} = X$. Now we have to show that $\mu_V = \lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee \cdots \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}$ and $\nu_V = \xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \cdots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}$. Let $x \in X$ and $\mu_V(x) = \lambda_j$ and then $\nu_V(x) = \xi_j$. Then if $\lambda_{j-1} > \lambda_j$ and $\xi_{j-1} = \xi_j$, $x \in \mu_V^{[\lambda_j]}$, $x \notin \mu_V^{[\lambda_{j-1}]}$ and $x \in \nu_V^{[\xi_j]}$ and $\nu_V^{[\xi_{j-1}]}$.
Then $\left(\lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee ... \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}\right)(x) = \lambda_j \vee \lambda_{j+1} \cdots \vee \lambda_k = \lambda_j$ and
$\left(\xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \cdots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}\right)(x) = \xi_{j-1} \wedge \xi_j \wedge \xi_{j+1} \wedge \cdots \wedge \xi_k = \xi_j$.
Similarly if $\lambda_{j-1} = \lambda_j$ and $\xi_{j-1} < \xi_j$ or if $\lambda_{j-1} > \lambda_j$ and $\xi_{j-1} < \xi_j$, then also
$\left(\lambda_0 1_{\mu_V^{[\lambda_0]}} \vee \lambda_1 1_{\mu_V^{[\lambda_1]}} \vee \cdots \vee \lambda_k 1_{\mu_V^{[\lambda_k]}}\right)(x) = \lambda_j$ and $\left(\xi_0 1_{\nu_V^{[\xi_0]}} \wedge \xi_1 1_{\nu_V^{[\xi_1]}} \wedge \cdots \wedge \xi_k 1_{\nu_V^{[\xi_k]}}\right)(x) = \xi_j$.
(1) and (2) are straightforward. □

Example 3.21. *Suppose* $X = \mathbb{R}^4$. *Define an intuitionistic fuzzy vector space* V *with* μ_V *and* ν_V *as follows:*
$\mu_V((0,0,0,0)) = .8$; $\mu_V((0,0,0,\mathbb{R}\setminus\{0\})) = .7$; $\mu_V((0,0,\mathbb{R}\setminus\{0\},\mathbb{R})) = .6$, $\mu_V((0,\mathbb{R}\setminus\{0\},\mathbb{R},\mathbb{R})) = .4$, $\mu_V(\mathbb{R}^4\setminus(0,\mathbb{R},\mathbb{R},\mathbb{R})) = .3$ *and* $\nu_V((0,0,0,0)) = .1$; $\nu_V((0,0,0,\mathbb{R}\setminus\{0\})) = .2$; $\nu_V((0,0,\mathbb{R}\setminus\{0\},\mathbb{R})) = .3$, $\nu_V((0,\mathbb{R}\setminus\{0\},\mathbb{R},\mathbb{R})) = .4$, $\nu_V(\mathbb{R}^4\setminus(0,\mathbb{R},\mathbb{R},\mathbb{R})) = .5$. *Then* $\mu_V = (.8)1_{\mu_V^{[.8]}} \vee (.7)1_{\mu_V^{[.7]}} \vee (.6)1_{\mu_V^{[.6]}} \vee (.4)1_{\mu_V^{[.4]}} \vee (.3)1_{\mu_V^{[.3]}}$ *and* $\nu_V = (.1)1_{\nu_V^{[.1]}} \wedge (.2)1_{\nu_V^{[.2]}} \wedge (.3)1_{\nu_V^{[.3]}} \wedge (.4)1_{\nu_V^{[.4]}} \wedge (.5)1_{\nu_V^{[.5]}}$.

Definition 3.22. *Let* $V \in$ *IFVS*(X) *with* $dim\ X = m$. *Consider Theorem 3.20. Let* B_{V_i} *be the basis of* $V^{[\lambda_i,\xi_i]}, i = 0,1,..,k$ *such that* $B_{V_0} \subsetneqq B_{V_1} \subsetneqq \cdots \subsetneqq B_{V_k}$ *(*).*
If $V^{(\lambda_0,\xi_0)} = \{\theta\}$, *we start with* $V^{(\lambda_1,\xi_1)}$.
Define a map $\mathbb{B}$ *from* X *to* $[0,1]\times[0,1]$ *by*
$$\mu_{\mathbb{B}}(x) = \begin{cases} \vee\{\lambda_i : x \in B_{V_i}\} \\ 0,\ otherwise \end{cases} \quad and\ \nu_{\mathbb{B}}(x) = \begin{cases} \vee\{\xi_i : x \in B_{V_i}\} \\ 1,\ otherwise \end{cases}.$$
Let $\mu_{\mathbb{B}}(x) = \lambda_j$. *Then* $x \in B_{V_j}$ *and* $x \notin B_{V_{j-1}}$ *i.e.* $x \in V^{[\lambda_j,\xi_j]}$ *and* $x \notin V^{[\lambda_{j-1},\xi_{j-1}]}$. *Thus* $\mu_V(x) \geq \lambda_j$ *and* $\nu_V(x) \leq \xi_j$. *If* $\mu_V(x) > \lambda_j$, *then* $\mu_V(x) = \lambda_l$ *for some* $l < j$. *Then* $x \in V^{[\lambda_l,\xi_l]}$ *and* $\mu_{(B)}(x) = \lambda_l$, *which is a contradiction. Therefore* $\mu_V(x) = \lambda_j$. *Then* $\nu_V(x) = \xi_j$ *i.e.* $\nu_{\mathbb{B}}(x) = \xi_j$. *Therefore* $\mathbb{B}$ *is an intuitionistic fuzzy set and it is called intuitionistic fuzzy basis of* V *corresponding to* $(*)$.

Example 3.23. *Consider the intuitionistic fuzzy vector space as in Example 3.21, where* $(\lambda_0,\xi_0) = (.8,.1), (\lambda_1,\xi_1) = (.7,.2), (\lambda_2,\xi_2) = (.6,.3), (\lambda_3,\xi_3) = (.4,.4)$ *and* $(\lambda_4,\xi_4) = (.3,.5)$. *Let* $e_1 = (0,0,0,1), e_2 = (0,0,1,0), e_3 = (0,1,0,0)$ *and* $e_4 = (1,0,0,0)$ *and* $B_{V_1} = \{e_1\}, B_{V_2} =$

$\{e_1, e_2\}, B_{V_3} = \{e_1, e_2, e_3\}$ *and* $B_{V_4} = \{e_1, e_2, e_3, e_4\}$. *Then* $\mathbb{B}$ *is an intuitionistic fuzzy basis of* V *which is defined by:*

$$\mu_{\mathbb{B}}(x) = \begin{cases} .7, & if\ x = e_1 \\ .6 & if\ x = e_2 \\ .4 & if\ x = e_3 \\ .3 & if\ x = e_4 \\ 0, & otherwise \end{cases} \text{ and } \nu_{\mathbb{B}}(x) = \begin{cases} .2, & if\ x = e_1 \\ .3 & if\ x = e_2 \\ .4 & if\ x = e_3 \\ .5 & if\ x = e_4 \\ 1, & otherwise \end{cases}.$$

Proposition 3.24. *Let* $\mathbb{B}$ *be an intuitionistic fuzzy basis of* V *corresponding to* $(*)$ *of Definition 3.22. Then*

(1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1})$ *with* $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, *then* $\mathbb{B}^{[\zeta,\rho]} = \mathbb{B}^{[\eta,\sigma]} = B_{V_i}$.

(2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_i, \xi_{i+1}), (\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ *with* $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, *then* $\mathbb{B}^{[\zeta,\rho]} \supsetneqq \mathbb{B}^{[\eta,\sigma]}$.

(3) $\mathbb{B}^{[\lambda,\xi]}$ *is a basis of* $V^{[\lambda,\xi]}$ *for* $\lambda \in (0, 1], \xi \in [0, 1)$ *with* $\lambda + \xi \leq 1$.

4 Conclusion

In our future study we have a plan to develop further properties of intuitionistic fuzzy vector spaces. In topological setting, studies on intuitionistic fuzzy topological vector spaces with intuitionistic fuzzy gradation of openness is also another interesting problem to be dealt with.

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