Notes on Intuitionistic Fuzzy Sets
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Extended temporal intuitionistic fuzzy index matrices

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Abstract: The concept of an extended temporal intuitionistic fuzzy index matrix is introduced and some of its basic properties are discussed.
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1 Introduction

The present text was written in the beginning of 2014 and planned to be published in NIFS 20, 3. Latter, it was included in my book [4], but for my mistake, it was not included in this and in the next isses of NIFS. Some colleagues asked me for its fate and even that moment I saw my mistake. So, to be correct with the reader of my book [4], I decided to publish the paper in its original form (the first four sections) and to extend it with the some new results (Section 5) that are obtained after publishing of [4].

Here, we use all definitions related to the concept of an Index Matrix (IM) from [4], these for Intuitionistic Fuzzy Set (IFS) from [2] and for Intuitionistic fuzzy logics – from [5].

In [6], Evdokia Sotirova, Veselina Bureva, Anthony Shannon and the author introduced the definition of the object Temporal IFIM (TIFIM) by

$$A(\mathcal{T}) = [K, L, \mathcal{T}, \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle\}]$$

$$\equiv \left\{ \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1,\tau}, \nu_{k_1,l_1,\tau} \rangle & \langle \mu_{k_1,l_2,\tau}, \nu_{k_1,l_2,\tau} \rangle & \dots & \langle \mu_{k_1,l_n,\tau}, \nu_{k_1,l_n,\tau} \rangle \\ \vdots & \vdots & \vdots & \dots & \vdots \\ k_i & \langle \mu_{k_i,l_1,\tau}, \nu_{k_i,l_1,\tau} \rangle & \langle \mu_{k_i,l_2,\tau}, \nu_{k_i,l_2,\tau} \rangle & \dots & \langle \mu_{k_i,l_n,\tau}, \nu_{k_i,l_n,\tau} \rangle \\ \vdots & \vdots & \vdots & \dots & \vdots \\ k_m & \langle \mu_{k_m,l_1,\tau}, \nu_{k_m,l_1,\tau} \rangle & \langle \mu_{k_m,l_2,\tau}, \nu_{k_m,l_2,\tau} \rangle & \dots & \langle \mu_{k_m,l_n,\tau}, \nu_{k_m,l_n,\tau} \rangle \end{array} | \tau \in \mathcal{T} \right\},$$

where for every $\tau \in \mathcal{T}$, $1 \le i \le m, 1 \le j \le n$:

$$\mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}, \mu_{k_i,l_j,\tau} + \nu_{k_i,l_j,\tau} \in [0, 1].$$

Here, $\mathcal{T}$ is a some fixed temporal scale and τ is its element, i.e., a time-moment.

By analogy with [4], Chapter 2, we extend the concept of a TIFIM, defining an Extended TIFIM (ETIFIM), by:

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle\}]$$

$$\equiv \left\{ \begin{array}{l|ccc} & l_1, \langle \alpha^l_{1,\tau}, \beta^l_{1,\tau} \rangle & \dots & l_n, \langle \alpha^l_{n,\tau}, \beta^l_{n,\tau} \rangle \\ \hline k_1, \langle \alpha^k_{1,\tau}, \beta^k_{1,\tau} \rangle & \langle \mu_{k_1,l_1,\tau}, \nu_{k_1,l_1,\tau} \rangle & \dots & \langle \mu_{k_1,l_n,\tau}, \nu_{k_1,l_n,\tau} \rangle \\ \vdots & \vdots & \dots & \vdots \\ k_i, \langle \alpha^k_{i,\tau}, \beta^k_{i,\tau} \rangle & \langle \mu_{k_i,l_1,\tau}, \nu_{k_i,l_1,\tau} \rangle & \dots & \langle \mu_{k_i,l_n,\tau}, \nu_{k_i,l_n,\tau} \rangle \\ \vdots & \vdots & \dots & \vdots \\ k_m, \langle \alpha^k_{m,\tau}, \beta^k_{m,\tau} \rangle & \langle \mu_{k_m,l_1,\tau}, \nu_{k_m,l_1,\tau} \rangle & \dots & \langle \mu_{k_m,l_n,\tau}, \nu_{k_m,l_n,\tau} \rangle \end{array} | \tau \in \mathcal{T} \right\},$$

where for every $1 \le i \le m, 1 \le j \le n$:

$$\mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}, \mu_{k_i,l_j,\tau} + \nu_{k_i,l_j,\tau} \in [0, 1],$$
$$\alpha^k_{i,\tau}, \beta^k_{i,\tau}, \alpha^k_{i,\tau} + \beta^k_{i,\tau} \in [0, 1],$$
$$\alpha^l_{j,\tau}, \beta^l_{j,\tau}, \alpha^l_{j,\tau} + \beta^l_{j,\tau} \in [0, 1]$$

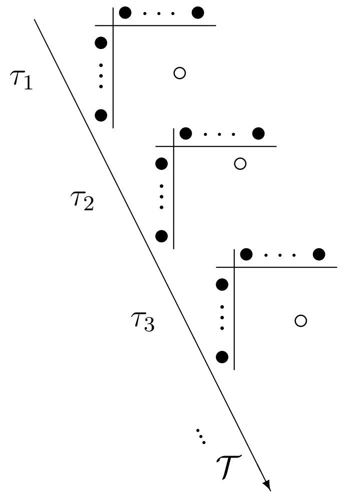
and

$$K^*(\mathcal{T}) = \{\langle k_i, \alpha^k_{i,\tau}, \beta^k_{i,\tau} \rangle | k_i \in K \;\&\; \tau \in \mathcal{T}\}$$
$$= \{\langle k_i, \alpha^k_{i,\tau}, \beta^k_{i,\tau} \rangle | 1 \le i \le m \;\&\; \tau \in \mathcal{T}\},$$
$$L^*(\mathcal{T}) = \{\langle l_j, \alpha^l_{j,\tau}, \beta^l_{j,\tau} \rangle | l_j \in L \;\&\; \tau \in \mathcal{T}\}$$
$$= \{\langle l_j, \alpha^l_{j,\tau}, \beta^l_{j,\tau} \rangle | 1 \le j \le n \;\&\; \tau \in \mathcal{T}\}.$$

Let

$$K^*(\mathcal{T}) \subset P^*(\mathcal{T}) \text{ iff } (K \subset P) \;\&\; (\forall \tau \in \mathcal{T})$$
$$(\forall k_i = p_i \in K : (\alpha^k_{i,\tau} < \alpha^p_{i,\tau}) \;\&\; (\beta^k_{i,\tau} > \beta^p_{i,\tau})).$$
$$K^*(\mathcal{T}) \subseteq P^*(\mathcal{T}) \text{ iff } (K \subseteq P) \;\&\; (\forall \tau \in \mathcal{T})$$
$$(\forall k_i = p_i \in K : (\alpha^k_{i,\tau} \le \alpha^p_{i,\tau}) \;\&\; (\beta^k_{i,\tau} \ge \beta^p_{i,\tau})).$$

We must mention that the new IM is 3-dimensional by analogy with the IM described in [4], Chapter 6. The indices for the third dimension of the TIFIM and ETIFIM are elements of a time-scale $\mathcal{T}$:



Below, we discuss the operations, relations and operators over the ETIFIMs.

2 Operations over ETIFIMs

For the ETIFIMs

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle\}],$$

$$B^*(\mathcal{T}) = [P^*(\mathcal{T}), Q^*(\mathcal{T}), \{\langle \rho_{p_r,q_s,\tau}, \sigma_{p_r,q_s,\tau}\rangle\}],$$

and for $(\circ, *) \in \{(\max, \min), (\min, \max)\}$, operations are the following.

Addition

$$A^*(\mathcal{T}) \oplus_{(\circ,*)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle\}],$$

where

$$T^*(\mathcal{T}) = K^*(\mathcal{T}) \cup P^*(\mathcal{T}) = \{\langle t_u, \alpha^t_{u,\tau}, \beta^t_{u,\tau}\rangle | t_u \in K \cup P \;\&\; \tau \in \mathcal{T}\},$$

$$V^*(\mathcal{T}) = L^*(\mathcal{T}) \cup Q^*(\mathcal{T}) = \{\langle v_w, \alpha^v_{w,\tau}, \beta^v_{w,\tau}\rangle | v_w \in L \cup Q \;\&\; \tau \in \mathcal{T}\},$$

$$\alpha^t_{u,\tau} = \begin{cases} \alpha^k_{i,\tau}, & \text{if } t_u \in K - P \\ \alpha^p_{r,\tau}, & \text{if } t_u \in P - K \\ \max(\alpha^k_{i,\tau}, \alpha^p_{r,\tau}), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta^v_{w,\tau} = \begin{cases} \beta^l_{j,\tau}, & \text{if } v_w \in L - Q \\ \beta^q_{s,\tau}, & \text{if } t_w \in Q - L \\ \min(\beta^l_{j,\tau}, \beta^q_{s,\tau}), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ \langle \rho_{p_r,q_s,\tau}, \sigma_{p_r,q_s,\tau}\rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ \langle \circ(\mu_{k_i,l_j,\tau}, \rho_{p_r,q_s,\tau}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad *(\nu_{k_i,l_j,t}, \sigma_{p_r,q_s,t})\rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases},$$

Termwise multiplication

$$A^*(\mathcal{T}) \otimes_{(\circ,*)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle\}],$$

where

$$T^*(\mathcal{T}) = K^*(\mathcal{T}) \cap P^*(\mathcal{T}) = \{\langle t_u, \alpha^t_{u,\tau}, \beta^t_{u,\tau}\rangle | t_u \in K \cap P \;\&\; \tau \in \mathcal{T}\},$$

$$V^*(\mathcal{T}) = L^*(\mathcal{T}) \cap Q^*(\mathcal{T}) = \{\langle v_w, \alpha^v_{w,\tau}, \beta^v_{w,\tau}\rangle | v_w \in L \cap Q \;\&\; \tau \in \mathcal{T}\},$$

$$\alpha^t_{u,\tau} = \min(\alpha^k_{i,\tau}, \alpha^p_{r,\tau}), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta^v_{w,\tau} = \max(\beta^l_{j,\tau}, \beta^q_{s,\tau}), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle = \langle \circ(\mu_{k_i,l_j,\tau}, \rho_{p_r,q_s,\tau}), *(\nu_{k_i,l_j,\tau}, \sigma_{p_r,q_s,\tau})\rangle.$$

Multiplication

$$A^*(\mathcal{T}) \odot_{(\circ,*)} B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle\}],$$

where

$$T^*(\mathcal{T}) = (K \cup (P - L))^*(\mathcal{T}) = \{\langle t_u, \alpha^t_{u,\tau}, \beta^t_{u,\tau}\rangle | t_u \in K \cup (P - L)\},$$

$$V^*(\mathcal{T}) = (Q \cup (L - P))^*(\mathcal{T}) = \{\langle v_w, \alpha^v_{w,\tau}, \beta^v_{w,\tau}\rangle | v_w \in Q \cup (L - P)\},$$

$$\alpha^t_{u,\tau} = \begin{cases} \alpha^k_{i,\tau}, & \text{if } t_u = k_i \in K \\ \\ \alpha^p_{r,\tau}, & \text{if } t_u = p_r \in P - L \end{cases},$$

$$\beta^v_{w,\tau} = \begin{cases} \beta^l_{j,\tau}, & \text{if } v_w = l_j \in L - P \\ \\ \beta^q_{s,\tau}, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau}\rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \\ \\ \langle \rho_{p_r,q_s,\tau}, \sigma_{p_r,q_s,\tau} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q \\ \\ \langle \underset{l_j = p_r \in L \cap P}{\circ} (\min(\mu_{k_i,l_j,\tau}, \rho_{p_r,q_s,\tau})), & \\ \\ \underset{l_j = p_r \in L \cap P}{*} (\max(\nu_{k_i,l_j,\tau}, \sigma_{p_r,q_s,\tau})) \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Structural subtraction

$$A^*(\mathcal{T}) \ominus B^*(\mathcal{T}) = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau} \rangle\}],$$

where

$$T^*(\mathcal{T}) = (K - P)^*(\mathcal{T}) = \{\langle t_u, \alpha^t_{u,\tau}, \beta^t_{u,\tau} \rangle | t_u \in K - P\},$$

$$V^*(\mathcal{T}) = (L - Q)^*(\mathcal{T}) = \{\langle v_w, \alpha^v_{w,\tau}, \beta^v_{w,\tau} \rangle | v_w \in L - Q\},$$

for the set–theoretic subtraction operation and

$$\alpha^t_{u,\tau} = \alpha^k_{i,\tau}, \text{ for } t_u = k_i \in K - P,$$

$$\beta^v_{w,\tau} = \beta^l_{j,\tau}, \text{ for } v_w = l_j \in L - Q$$

and

$$\langle \varphi_{t_u,v_w,\tau}, \psi_{t_u,v_w,\tau} \rangle = \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle,$$

for $t_u = k_i \in K - P$ and $v_w = l_j \in L - Q$.

Negation of an ETIFIM

$$\neg A(\mathcal{T})^* = [T^*(\mathcal{T}), V^*(\mathcal{T}), \{\neg \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle\}],$$

where $\neg$ is one of the negations that can be defined over intuitionisitc fuzzy

Termwise subtraction

$$A^*(\mathcal{T}) -_{(\circ,*)} B^*(\mathcal{T}) = A^*(\mathcal{T}) \oplus_{(\circ,*)} \neg B^*(\mathcal{T}).$$

Operations “reduction”, “projection” and “substitution” coincide with the respective operations defined over IMs in [4], while the hierarchical operations over IMs are not applicable here.

3 Relations over ETIFIMs

Let the two ETIFIMs

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle\}]$$

and

$$B^*(\mathcal{T}) = [P^*(\mathcal{T}), Q^*(\mathcal{T}), \{\langle \rho_{p_r,q_s,\tau}, \sigma_{p_r,q_s,\tau} \rangle\}]$$

be given. We introduce the following definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.

The strict relation "inclusion about matrix-dimension and elements" is

$$A^*(\mathcal{T}) \subset_d^e B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee\, ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee\, (K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T})))$$

$$\&\, (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle = \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau} \rangle).$$

The non-strict relation "inclusion about matrix-dimension and elements" is

$$A^*(\mathcal{T}) \subseteq_d^e B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))$$

$$\&\, (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle = \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau} \rangle).$$

The strict relation "inclusion about element values" is

$$A^*(\mathcal{T}) \subset_v^e B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) = Q^*(\mathcal{T}))$$

$$\&\, (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle < \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau} \rangle).$$

The non-strict relation "inclusion about element values" is

$$A^*(\mathcal{T}) \subseteq_v^e B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) = Q^*(\mathcal{T}))$$

$$\&\, (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle \leq \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau} \rangle).$$

The strict relation "inclusion about matrix-dimension and element values" is

$$A^*(\mathcal{T}) \subset^e B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee\, ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee\, ((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subseteq Q)^*(\mathcal{T})))$$

$$\&\, (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau} \rangle < \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau} \rangle).$$

The non-strict relation "inclusion about matrix-dimension and element values" is

$$A^*(\mathcal{T}) \subseteq^e B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \,\&\, (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))$$

$$\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)(\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle \leq \langle \rho_{k_i,l_j,\tau}, \sigma_{k_i,l_j,\tau}\rangle).$$

The strict relation "inclusion about matrix-dimension and indices" is

$$A^*(\mathcal{T}) \subset_d^i B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T}))) $$

$$\vee \ ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee \ ((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))))\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle = \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle = \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

The non-strict relation "inclusion about matrix-dimension and indices" is

$$A^*(\mathcal{T}) \subseteq_d^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))$$

$$\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle = \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle = \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

The strict relation "inclusion about matrix-dimension and index values" is

$$A^*(\mathcal{T}) \subset_v^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T}))$$

$$\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle < \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle < \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

The non-strict relation "inclusion about matrix-dimension and index values" is

$$A^*(\mathcal{T}) \subseteq_v^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) = P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) = Q^*(\mathcal{T}))$$

$$\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle \leq \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle \leq \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

The strict relation "inclusion about index values" is

$$A^*(\mathcal{T}) \subset^i B^*(\mathcal{T}) \text{ iff } (((K^*(\mathcal{T}) \subset P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee \ ((K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subset Q^*(\mathcal{T})))$$

$$\vee \ ((K^*(\mathcal{T}) \subset P^*(\mathcal{T}))\&(L^*(\mathcal{T}) \subseteq Q)^*(\mathcal{T}))) \ \& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle < \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle < \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

The non-strict relation "inclusion about index values" is

$$A^*(\mathcal{T}) \subseteq^i B^*(\mathcal{T}) \text{ iff } (K^*(\mathcal{T}) \subseteq P^*(\mathcal{T})) \ \& \ (L^*(\mathcal{T}) \subseteq Q^*(\mathcal{T}))$$

$$\& \ (\forall \tau \in (\mathcal{T}))(\forall k \in K)(\forall l \in L)$$

$$(\langle \alpha_{i,\tau}^k, \beta_{i,\tau}^k\rangle \leq \langle \alpha_{i,\tau}^p, \beta_{i,\tau}^p\rangle \ \& \ \langle \alpha_{i,\tau}^l, \beta_{i,\tau}^l\rangle \leq \langle \alpha_{i,\tau}^q, \beta_{i,\tau}^q\rangle).$$

ETIFIM $A^*(\mathcal{T})$ has temporal strictly increasing elements, if

$$(\forall \tau_1, \tau_2 \in \mathcal{T})((\tau_1 < \tau_2) \to (\forall k_i \in K)(\forall l_j \in L)$$

$$(\langle \mu_{k_i,l_j,\tau_1}, \nu_{k_i,l_j,\tau_1}\rangle < \langle \rho_{k_i,l_j,\tau_2}, \sigma_{k_i,l_j,\tau_2}\rangle).$$

ETIFIM $A^*(\mathcal{T})$ has temporal non-strictly increasing elements, if

$$(\forall \tau_1, \tau_2 \in \mathcal{T})((\tau_1 < \tau_2) \to (\forall k_i \in K)(\forall l_j \in L)$$

$$(\langle \mu_{k_i,l_j,\tau_1}, \nu_{k_i,l_j,\tau_1}\rangle \leq \langle \rho_{k_i,l_j,\tau_2}, \sigma_{k_i,l_j,\tau_2}\rangle).$$

ETIFIM $A^*(\mathcal{T})$ has temporal strictly increasing indices, if

$$(\forall \tau_1, \tau_2 \in \mathcal{T})((\tau_1 < \tau_2) \to (\forall k_i \in K)(\forall l_j \in L)$$

$$(\langle \alpha^k_{i,\tau_1}, \beta^k_{i,\tau_1}\rangle < \langle \alpha^p_{i,\tau_2}, \beta^p_{i,\tau_2}\rangle \ \&\ \langle \alpha^l_{j,\tau_1}, \beta^l_{j,\tau_1}\rangle < \langle \alpha^q_{j,\tau_2}, \beta^q_{j,\tau_2}\rangle).$$

ETIFIM $A^*(\mathcal{T})$ has temporal non-strictly increasing indices, if

$$(\forall \tau_1, \tau_2 \in \mathcal{T})((\tau_1 < \tau_2) \to (\forall k_i \in K)(\forall l_j \in L)$$

$$(\langle \alpha^k_{i,\tau_1}, \beta^k_{i,\tau_1}\rangle \leq \langle \alpha^p_{i,\tau_2}, \beta^p_{i,\tau_2}\rangle \ \&\ \langle \alpha^l_{j,\tau_1}, \beta^l_{j,\tau_1}\rangle \leq \langle \alpha^q_{j,\tau_2}, \beta^q_{j,\tau_2}\rangle).$$

4 Specific operations over ETIFIMs

Let operations $\overline{\circ}$ and $\overline{*}$ are dual operations of operations $\circ$ and $*$, respectively. For example, pairs $(\circ, \overline{\circ})$ and $(*, \overline{*})$ can be any among pairs $(\max, \min)$, $(\min, \max)$. If $\circ$ and $*$ are average operations, then $\overline{\circ}$ and $\overline{*}$ are also average operations.

Let the time-scale $\mathcal{T}$ be fixed, let the ETIFIM

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle\}]$$

$$\equiv \left\{ \begin{array}{c|ccc} & l_1, \langle \alpha^l_{1,\tau}, \beta^l_{1,\tau}\rangle & \dots & l_n, \langle \alpha^l_{n,\tau}, \beta^l_{n,\tau}\rangle \\ \hline k_1, \langle \alpha^k_{1,\tau}, \beta^k_{1,\tau}\rangle & \langle \mu_{k_1,l_1,\tau}, \nu_{k_1,l_1,\tau}\rangle & \dots & \langle \mu_{k_1,l_n,\tau}, \nu_{k_1,l_n,\tau}\rangle \\ \vdots & \vdots & \dots & \vdots \\ k_m, \langle \alpha^k_{m,\tau}, \beta^k_{m,\tau}\rangle & \langle \mu_{k_m,l_1,\tau}, \nu_{k_m,l_1,\tau}\rangle & \dots & \langle \mu_{k_m,l_n,\tau}, \nu_{k_m,l_n,\tau}\rangle \end{array} |\tau \in \mathcal{T} \right\}$$

be given and let $k_0 \notin K$ and $l_0 \notin L$ be two indices. Now, we introduce the following operations over it:

$(\circ, *)$**-row-aggregation**

$$\rho_{(\circ,*)}(A^*(\mathcal{T}), k_0)$$

$$= \left\{ \begin{array}{c|cc} & l_1, \langle \alpha^l_{1,\tau}, \beta^l_{1,\tau}\rangle & \dots \\ \hline k_0, \langle \underset{1\leq i\leq m}{\circ} \alpha^k_{i,\tau}, \underset{1\leq i\leq m}{\overline{\circ}} \beta^k_{i,\tau}\rangle & \langle \underset{1\leq i\leq m}{*} \mu_{k_i,l_1,\tau}, \underset{1\leq i\leq m}{\overline{*}} \nu_{k_i,l_1,\tau}\rangle & \dots \end{array} \right.$$

$$
\begin{array}{c|c}
\ldots & l_n, \langle \alpha^l_{n,\tau}, \beta^l_{n,\tau} \rangle \\
\hline
\ldots & \langle \underset{1\le i\le m}{*} \mu_{k_i,l_n,\tau}, \underset{1\le i\le m}{\overline{*}} \nu_{k_i,l_n,\tau} \rangle
\end{array}
|\tau \in \mathcal{T} \Bigg\},
$$

$(\circ, *)$**-column-aggregation**

$$
\sigma_{(\circ,*)}(A^*(\mathcal{T}), l_0)
$$

$$
= \left\{
\begin{array}{c|c}
 & l_0, \langle \underset{1\le j\le n}{\circ} \alpha^l_{j,\tau}, \underset{1\le j\le n}{\overline{\circ}} \beta^l_{j,\tau} \rangle \\
\hline
k_1, \langle \alpha^k_{1,\tau}, \beta^k_{1,\tau} \rangle & \langle \underset{1\le j\le n}{*} \mu_{k_1,l_j,\tau}, \underset{1\le j\le n}{\overline{*}} \nu_{k_i,l_j,\tau} \rangle \\
\vdots & \vdots \\
k_m, \langle \alpha^k_{m,\tau}, \beta^k_{m,\tau} \rangle & \langle \underset{1\le j\le n}{*} \mu_{k_m,l_j,\tau}, \underset{1\le j\le n}{\overline{*}} \nu_{k_i,l_j,\tau} \rangle
\end{array}
|\tau \in \mathcal{T} \right\}.
$$

Let $\bullet$ and $\overline{\bullet}$ be two other dual operations, that can coincide with $(\circ, \overline{\circ})$ or $(*, \overline{*})$, or to be different. The following new aggregation operation can be defined:

$(\circ, \bullet, *) - \mathcal{T}$**-aggregation**

$$
\rho^{\mathcal{T}}_{(\circ,\bullet,*)}(A^*(\mathcal{T}))
$$

$$
= \begin{array}{c|cc}
 & l_1, \langle \underset{\tau\in\mathcal{T}}{\bullet} \alpha^l_{1,\tau}, \underset{\tau\in\mathcal{T}}{\overline{\bullet}} \beta^l_{1,\tau} \rangle & \ldots \\
\hline
k_1, \langle \underset{\tau\in\mathcal{T}}{\circ} \alpha^k_{1,\tau}, \underset{\tau\in\mathcal{T}}{\overline{\circ}} \beta^k_{1,\tau} \rangle & \langle \underset{\tau\in\mathcal{T}}{*} \mu_{k_1,l_1,\tau}, \underset{\tau\in\mathcal{T}}{\overline{*}} \nu_{k_1,l_1,\tau} \rangle & \ldots \\
\vdots & \vdots & \ldots \\
k_m, \langle \underset{\tau\in\mathcal{T}}{\circ} \alpha^k_{m,\tau}, \underset{\tau\in\mathcal{T}}{\overline{\circ}} \beta^k_{m,\tau} \rangle & \langle \underset{\tau\in\mathcal{T}}{*} \mu_{k_m,l_1,\tau}, \underset{\tau\in\mathcal{T}}{\overline{*}} \nu_{k_m,l_1,\tau} \rangle & \ldots
\end{array}
$$

$$
\begin{array}{cc}
\ldots & l_n, \langle \alpha^l_{n,\tau}, \beta^l_{n,\tau} \rangle \\
\hline
\ldots & \langle \underset{\tau\in\mathcal{T}}{*} \mu_{k_1,l_n,\tau}, \underset{\tau\in\mathcal{T}}{\overline{*}} \nu_{k_1,l_n,\tau} \rangle \\
\vdots & \vdots \\
\ldots & \langle \underset{\tau\in\mathcal{T}}{*} \mu_{k_m,l_n,\tau}, \underset{\tau\in\mathcal{T}}{\overline{*}} \nu_{k_m,l_n,\tau} \rangle \rangle
\end{array}
.
$$

The above operations are extensions of the operations from IFIMs, EIFIMs and TIFIMs and have similar properties.

5 Level operators over EIFIMs

Now, we extend the idea from [3], introducing level operators over ETIFIMs.

Let the ETIFIM

$$A^*(\mathcal{T}) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle\}]$$

be given.

Let for $i = 1, 2, 3 : \rho_i, \sigma_i, \rho_i + \sigma_i \in [0, 1]$ be fixed.

In [1, 2], some level operators are defined. One of them, for a given IFS

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E\}$$

is

$$N_{\alpha,\beta}(X) = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E \ \&\ \mu_X(x) \geq \alpha \ \&\ \nu_X(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ are fixed and $\alpha + \beta \leq 1$.

Here, its analogues are intoduced. They are three: $N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3}$ and influence over $K^*(\mathcal{T})$-, $L^*(\mathcal{T})$-indices and $\langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle$-elements, respectively. The three operators can be applied over an ETIFIM A sequentially, or simultaneously. In the first case, theis forms are

$$N^1_{\rho_1,\sigma_1}(A^*(\mathcal{T})) = [N_{\rho_1,\sigma_1}(K^*(\mathcal{T})), L^*(\mathcal{T}), \{\langle \varphi_{k_i,l_j,\tau}, \psi_{k_i,l_j,\tau}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j,\tau}, \psi_{k_i,l_j,\tau}\rangle = \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle$$

only for $\langle k_i, \alpha^k_i, \beta^k_i\rangle \in N_{\rho_1,\sigma_1}(K^*(\mathcal{T}))$ and for each $\langle l_j, \alpha^l_j, \beta^l_j\rangle \in L^*(\mathcal{T})$;

$$N^2_{\rho_2,\sigma_2}(A^*(\mathcal{T})) = [K^*(\mathcal{T}), N_{\rho_2,\sigma_2}(L^*(\mathcal{T})), \{\langle \varphi_{k_i,l_j,\tau}, \psi_{k_i,l_j,\tau}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j,\tau}, \psi_{k_i,l_j,\tau}\rangle = \langle \mu_{k_i,l_j,\tau}, \nu_{k_i,l_j,\tau}\rangle$$

for each $\langle k_i, \alpha^k_i, \beta^k_i\rangle \in K^*(\mathcal{T})$ and only for $\langle l_j, \alpha^l_j, \beta^l_j\rangle \in N_{\rho_2,\sigma_2}(L^*(\mathcal{T}))$;

$$N^3_{\rho_3,\sigma_3}(A^*(\mathcal{T})) = [K^*(\mathcal{T}), L^*(\mathcal{T}), \{\langle \varphi_{k_i,l_j,\tau}, \psi_{k_i,l_j\tau}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j\tau}, \psi_{k_i,l_j\tau}\rangle = \begin{cases} \langle \mu_{k_i,l_j\tau}, \nu_{k_i,l_j\tau}\rangle, & \text{if } \mu_{k_i,l_j\tau} \geq \rho_3 \ \&\ \nu_{k_i,l_j\tau} \leq \sigma_3 \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases},$$

In the second case, their form is

$$(N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3})(A^*(\mathcal{T})) = [N_{\rho_1,\sigma_1}(K^*(\mathcal{T})), N_{\rho_2,\sigma_2}(L^*(\mathcal{T})), \{\langle \varphi_{k_i,l_j\tau}, \psi_{k_i,l_j\tau}\rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle = \begin{cases} \langle \mu_{k_i,l_j\mathcal{T}}, \nu_{k_i,l_j\mathcal{T}} \rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k \rangle \in N_{\rho_1,\sigma_1}(K^*(\mathcal{T})) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l \rangle \in N_{\rho_2,\sigma_2}(L^*(\mathcal{T})) \\ & \text{and } \mu_{k_i,l_j\mathcal{T}} \geq \rho_3 \;\&\; \nu_{k_i,l_j\mathcal{T}} \leq \sigma_3 \\ \\ \langle 0, 1 \rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k \rangle \in N_{\rho_1,\sigma_1}(K^*(\mathcal{T})) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l \rangle \in N_{\rho_2,\sigma_2}(L^*(\mathcal{T})) \\ & \text{and } \mu_{k_i,l_j\mathcal{T}} < \rho_3 \;\vee\; \nu_{k_i,l_j\mathcal{T}} > \sigma_3 \end{cases},$$

6 Conclusion

As we mentioned above, the newly defined type of IMs is 3-dimensional. In future, we will introduce new opertions over the ETIFIMs. These operations will be specific for the ETIFIM and will do not have analogous as in matrix theory as well as in IM-theory.

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References

[1] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets*. Springer, Heidelberg.

[2] Atanassov, K. (2102) *On Intuitionistic Fuzzy Sets*, Springer, Berlin.

[3] Atanassov, K. (2013) On extended intuitionistic fuzzy index matrices, *Notes on Intuitionistic Fuzzy Sets*, 19(4), 27–41.

[4] Atanassov, K. (2014) *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham.

[5] Atanassov, K. (2017) *Intuitionistic Fuzzy Logics*, Springer, Cham (in press).

[6] Atanassov, K., Sotirova, E., Bureva, V. & Shannon, A. (2013) Temporal intuitionistic fuzzy index matrices. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, 10, 54–65.