Upper and lower limits and $\mathbf{m}$ -almost everywhere convergence of intuitionistic fuzzy observables

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*This paper is dedicated to the memory
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Abstract: In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables with the help of intuitionistic fuzzy probability $\mathcal{P}$ and we compared two concepts of $\mathcal{P}$ -almost everywhere convergence. The aim of this paper is to define the lower and upper limits using the intuitionistic fuzzy state $\mathbf{m}$. We study two concepts of $\mathbf{m}$ -almost everywhere convergence and we show that they are equivalent, too.

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1 Introduction

In [1, 2] K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [3] Grzegorzewski and Mrówka defined the probability on the family of intuitionistic fuzzy events

$$\mathcal{N} = \{(\mu_A, \nu_A) ; \mu_A + \nu_A \leq 1\},$$

where μ_A, ν_A are $\mathcal{S}$ -measurable, as a mapping $\mathcal{P}$ from the family $\mathcal{N}$ to the set of all compact intervals in $\mathbb{R}$ by the formula

$$\mathcal{P}((\mu_A, \nu_A)) = \left[\int_{\Omega} \mu_A dP, 1 - \int_{\Omega} \nu_A dP \right],$$

where $(\Omega, \mathcal{S}, P)$ is probability space. This IF-probability was axiomatically characterized by B. Riečan (see[5]).

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables with the help of intuitionistic fuzzy probability $\mathcal{P}$ and we compared two concepts of $\mathcal{P}$ -almost everywhere convergence, where $\mathbf{m}$ is an intuitionistic fuzzy state. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposed to two intuitionistic fuzzy states, it is useful to study $\mathbf{m}$ -almost everywhere convergence. In this paper we define the lower and upper limits for a sequence of intuitionistic fuzzy observables and we study two concepts of $\mathbf{m}$ -almost everywhere convergence, too.

Remark that in a whole text we use a notation “IF” for short a phrase “intuitionistic fuzzy”.

2 IF-events, IF-states and IF-observables

Definition 2.1. Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \rightarrow [0, 1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.

Definition 2.2. Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ -algebra of subsets of Ω . An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \rightarrow [0, 1]$ are $\mathcal{S}$ -measurable.

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \rightarrow [0, 1]$ will be called the membership function, $\nu_A : \Omega \rightarrow [0, 1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Łukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1)\end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \leq \mathbf{B} \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In the paper we use the max-min connectives defined by

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$

$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the De Morgan rules

$$(a \vee b)^* = a^* \wedge b^*,$$

$$(a \wedge b)^* = a^* \vee b^*,$$

where $a^* = 1 - a$.

Example 2.3. The fuzzy set $f : \Omega \longrightarrow [0, 1]$ can be regarded as an IF-set, if we put

$$\mathbf{A} = (f, 1 - f).$$

If $f = \chi_A$, then the corresponding IF-set has the form

$$\mathbf{A} = (\chi_A, 1 - \chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the product of sets, and $\leq$ to the set inclusion.

In the IF-probability theory ([7], [8]) instead of the notion of probability we use the notion of state.

Definition 2.4. Let $\mathcal{F}$ be the family of all IF-events in Ω . A mapping $\mathbf{m} : \mathcal{F} \rightarrow [0, 1]$ is called an IF-state, if the following conditions are satisfied:

- (i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;
- (ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;
- (iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Probably the most useful result in the IF-state theory is the following representation theorem ([6]):

Theorem 2.5. To each IF-state $\mathbf{m} : \mathcal{F} \rightarrow [0, 1]$ there exists exactly one probability measure $P : \mathcal{S} \rightarrow [0, 1]$ and exactly one $\alpha \in [0, 1]$ such that

$$\mathbf{m}(\mathbf{A}) = (1 - \alpha) \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} \nu_A dP \right)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R \mid a \leq x < b\}.$$

Then the σ -algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ -algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \rightarrow \mathcal{F}$ satisfying the following conditions:

- (i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;
- (ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;
- (iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If we denote $x(A) = (x^b(A), 1 - x^\sharp(A))$ for each $A \in \mathcal{B}(R)$, then $x^b, x^\sharp : \mathcal{B}(R) \rightarrow \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \rightarrow [0, 1]; f \text{ is } \mathcal{S}\text{-measurable}\}$.

Similarly as in the classical case the following theorem can be proved ([8]).

Theorem 2.7. Let $x : \mathcal{B}(R) \rightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \rightarrow [0, 1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \rightarrow [0, 1]$ by the formula

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \rightarrow [0, 1]$ is a probability measure.

3 Lower and upper limits

Let $(\xi_n)_n$ be a sequence of random variables in a probability space $(\Omega, \mathcal{S}, P)$. It is easy to see that

$$\limsup_{n \rightarrow \infty} \xi_n(\omega) < t \iff \omega \in \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \xi^{-1}\left(\left(-\infty, t - \frac{1}{p}\right)\right),$$

for every $\omega \in \Omega$. This relation justifies the following definition.

Definition 3.1. We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup_{n \rightarrow \infty}$, if there exists an IF-observable $\bar{x} : \mathcal{B}(R) \rightarrow \mathcal{F}$ such that

$$\bar{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)$$

for every $t \in R$. We write $\bar{x} = \limsup_{n \rightarrow \infty} x_n$.

Note that if another IF-observable y satisfies the above condition, then $\mathbf{m} \circ y = \mathbf{m} \circ \bar{x}$.

Definition 3.2. A sequence $(x_n)_n$ of IF-observables has $\liminf_{n \rightarrow \infty}$, if there exists an IF-observable $\underline{x}$ such that

$$\underline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)$$

for all $t \in R$. Notation: $\underline{x} = \liminf_{n \rightarrow \infty} x_n$.

Theorem 3.3. The IF-observables $\bar{x}$, $\underline{x}$ from Definition 3.1 and Definition 3.2 can be expressed in the following form

$$\begin{aligned} \bar{x}(A) &= \left(\overline{x^b}(A), 1 - \overline{x^\sharp}(A)\right), \\ \underline{x}(A) &= \left(\underline{x^b}(A), 1 - \underline{x^\sharp}(A)\right), \end{aligned}$$

for each $A \in \mathcal{B}(R)$. Here $\overline{x^b}$, $\underline{x^b}$ are upper and lower limits of sequence $(x_n^b)_1^\infty$ of observables in tribe $\mathcal{T}$ and $\overline{x^\sharp}$, $\underline{x^\sharp}$ are upper and lower limits of sequence $(x_n^\sharp)_1^\infty$ of observables in tribe $\mathcal{T}$ (see [9]).

Proof. Let $(x_n)_1^\infty$ be a sequence of IF-observables. Denote

$$x_n(A) = (x_n^b(A), 1 - x_n^\sharp(A))$$

for each $A \in \mathcal{B}(R)$, then $x_n^b, x_n^\sharp : \mathcal{B}(R) \rightarrow \mathcal{T}$ are observables for every $n \in N$.

Therefore using definition of max-min connectives $\vee, \wedge$ and the De Morgan rules, we obtain

$$\begin{aligned} \overline{x}(-\infty, t) &= \bigvee_{p=1}^\infty \bigvee_{k=1}^\infty \bigwedge_{n=k}^\infty x_n\left(-\infty, t - \frac{1}{p}\right) = \\ &= \left(\bigvee_{p=1}^\infty \bigvee_{k=1}^\infty \bigwedge_{n=k}^\infty x_n^b\left(-\infty, t - \frac{1}{p}\right), 1 - \bigvee_{p=1}^\infty \bigvee_{k=1}^\infty \bigwedge_{n=k}^\infty x_n^\sharp\left(-\infty, t - \frac{1}{p}\right) \right) = \\ &= \left(\overline{x^b}(-\infty, t), 1 - \overline{x^\sharp}(-\infty, t) \right) \end{aligned}$$

and

$$\begin{aligned} \underline{x}(-\infty, t) &= \bigvee_{p=1}^\infty \bigwedge_{k=1}^\infty \bigvee_{n=k}^\infty x_n\left(-\infty, t - \frac{1}{p}\right) = \\ &= \left(\bigvee_{p=1}^\infty \bigwedge_{k=1}^\infty \bigvee_{n=k}^\infty x_n^b\left(-\infty, t - \frac{1}{p}\right), 1 - \bigvee_{p=1}^\infty \bigwedge_{k=1}^\infty \bigvee_{n=k}^\infty x_n^\sharp\left(-\infty, t - \frac{1}{p}\right) \right) = \\ &= \left(\underline{x^b}(-\infty, t), 1 - \underline{x^\sharp}(-\infty, t) \right) \end{aligned}$$

for every $t \in R$. □

Proposition 3.1. *Let $(x_n)_n$ be a bounded sequence of IF-observables, i.e., there exist IF-observables $y, z : \mathcal{B}(R) \rightarrow \mathcal{F}$ such that*

$$y((-\infty, t)) \leq x_n((-\infty, t)) \leq z((-\infty, t))$$

for every $t \in R$ and every $n \in N$. Then there exists $\limsup_{n \rightarrow \infty} x_n$.

Proof. Since y, x_n, z are IF-observables, then we can write them in the following form

$$\begin{aligned} y((-\infty, t)) &= (y^b((-\infty, t)), 1 - y^\sharp((-\infty, t))), \\ x_n((-\infty, t)) &= (x_n^b((-\infty, t)), 1 - x_n^\sharp((-\infty, t))), \\ z((-\infty, t)) &= (z^b((-\infty, t)), 1 - z^\sharp((-\infty, t))), \end{aligned}$$

where $y^b, y^\sharp, x_n^b, x_n^\sharp, z^b, z^\sharp : \mathcal{B}(R) \rightarrow \mathcal{T}$ are observables.

Such

$$y((-\infty, t)) \leq x_n((-\infty, t)) \leq z((-\infty, t)),$$

then

$$(y^b((-\infty, t)), 1 - y^\sharp((-\infty, t))) \leq (x_n^b((-\infty, t)), 1 - x_n^\sharp((-\infty, t))) \leq (z^b((-\infty, t)), 1 - z^\sharp((-\infty, t))).$$

Hence

$$y^b((-\infty, t)) \leq x_n^b((-\infty, t)) \leq z^b((-\infty, t)),$$

$$1 - y^\sharp((-\infty, t)) \geq 1 - x_n^\sharp((-\infty, t)) \geq 1 - z^\sharp((-\infty, t)),$$

$$y^\sharp((-\infty, t)) \leq x_n^\sharp((-\infty, t)) \leq z^\sharp((-\infty, t))$$

for every $t \in R$ and every $n \in N$.

Using Proposition 8.6.2 from [9] then there exist upper limits $\overline{x^b}, \overline{x^\sharp} : \mathcal{B}(R) \rightarrow \mathcal{T}$. Finally, from Theorem 3.3 we obtain that there exists $\overline{x} = \limsup_{n \rightarrow \infty} x_n$ given by

$$\overline{x}((-\infty, t)) = \left(\overline{x^b}((-\infty, t)), 1 - \overline{x^\sharp}((-\infty, t)) \right). \quad \square$$

Proposition 3.2. *If a sequence of IF-observables $(x_n)_n$ has $\overline{x} = \limsup_{n \rightarrow \infty} x_n$ and $\underline{x} = \liminf_{n \rightarrow \infty} x_n$, then*

$$\overline{x}((-\infty, t)) \leq \underline{x}((-\infty, t)),$$

for every $t \in R$

Proof. Let a sequence of IF-observables $(x_n)_n$ has $\overline{x}, \underline{x}$. Then by Theorem 3.3 there exist observables $\overline{x^b}, \underline{x^b}, \overline{x^\sharp}, \underline{x^\sharp}$ in tribe $\mathcal{T}$, such that

$$\overline{x}((-\infty, t)) = \left(\overline{x^b}((-\infty, t)), 1 - \overline{x^\sharp}((-\infty, t)) \right),$$

$$\underline{x}((-\infty, t)) = \left(\underline{x^b}((-\infty, t)), 1 - \underline{x^\sharp}((-\infty, t)) \right).$$

Hence by Proposition 8.6.4 in [9] we have the following inequalities for every $t \in R$

$$\overline{x^b}((-\infty, t)) \leq \underline{x^b}((-\infty, t)), \quad (1)$$

$$\overline{x^\sharp}((-\infty, t)) \leq \underline{x^\sharp}((-\infty, t)),$$

then we obtain

$$1 - \overline{x^\sharp}((-\infty, t)) \geq 1 - \underline{x^\sharp}((-\infty, t)) \quad (2)$$

by simple modifications.

Therefore, by (1), (2) and Theorem 3.3 we have

$$\overline{x}((-\infty, t)) \leq \underline{x}((-\infty, t)). \quad \square$$

4 Almost everywhere convergence

Recall that a sequence $(\xi_n)_n$ of random variables converges almost everywhere to 0, if

$$P\left(\bigcap_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \xi_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = \lim_{n \rightarrow \infty} P\left(\bigcap_{n=k}^{k+i} \xi_n^{-1}\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

Similarly we can work with a sequence of IF-observables on $(\mathcal{F}, \mathbf{m})$.

Definition 4.1. *Let $(x_n)_n$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(x_n)_n$ converges $\mathbf{m}$ -almost everywhere to 0, if*

$$\mathbf{m}\left(\bigcap_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = \lim_{n \rightarrow \infty} \mathbf{m}\left(\bigcap_{n=k}^{k+i} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

Remark 4.2. The defining formula is equivalent to the following equality

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 0.$$

Definition 4.3. By the zero IF-observable $0_{\mathcal{F}}$ we shall denote the IF-observable defined by the following formula

$$0_{\mathcal{F}}((-\infty, t)) = \begin{cases} (1, 0), & \text{if } t > 0 \\ (0, 1), & \text{if } t \leq 0 \end{cases}$$

for each $t \in R$.

Remark 4.4. The zero IF-observable $0_{\mathcal{F}}$ can be rewritten in the following form

$$0_{\mathcal{F}}(A) = \begin{cases} (1, 0), & \text{if } 0 \in A \\ (0, 1), & \text{if } 0 \notin A \end{cases}$$

for each $A = (-\infty, t)$ and $t \in R$.

Proposition 4.1. A sequence $(x_n)_n$ of IF-observables converges $\mathbf{m}$ -almost everywhere to 0 if and only if

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}(0_{\mathcal{F}}((-\infty, t))),$$

for every $t \in R$.

Proof. “ $\Rightarrow$ ” Let a sequence $(x_n)_n$ of an IF-observables converges $\mathbf{m}$ -almost everywhere to 0. Then from Definition 4.1 we have

$$\mathbf{m}\left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = \lim_{n \rightarrow \infty} \mathbf{m}\left(\bigwedge_{n=k}^{k+i} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1, \quad (3)$$

i.e.,

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 0. \quad (4)$$

Since $R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right) = \left(-\infty, -\frac{1}{p}\right] \cup \left[\frac{1}{p}, \infty\right)$, then

$$x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right) = x_n\left(\left(-\infty, -\frac{1}{p}\right]\right) \oplus x_n\left(\left[\frac{1}{p}, \infty\right)\right).$$

For $t \leq 0$ is $\left(-\infty, t - \frac{1}{p}\right) \subset \left(-\infty, -\frac{1}{p}\right) \subset R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)$, hence

$$x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right) \leq x_n\left(\left(-\infty, -\frac{1}{p}\right)\right) \leq x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)$$

and therefore

$$\mathbf{m}\left(\bigvee_{n=k}^{k+i} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \leq \mathbf{m}\left(\bigvee_{n=k}^{k+i} x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right). \quad (5)$$

Finally from Proposition 3.2, (4) and (5) we obtain

$$\begin{aligned} 0 \leq \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) &\leq \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \leq \\ &\leq \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 0. \end{aligned}$$

But $\mathbf{m}(0_{\mathcal{F}}((-\infty, t))) = 0$ for $t \leq 0$, hence

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}(0_{\mathcal{F}}((-\infty, t))).$$

On the other hand, if $t > 0$, then there exists p such that $(-\frac{1}{p}, \frac{1}{p}) \subset (-\infty, t - \frac{1}{p})$. Therefore

$$x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right) \leq x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)$$

and hence

$$\mathbf{m}\left(\bigwedge_{n=k}^{k+i} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) \leq \mathbf{m}\left(\bigwedge_{n=k}^{k+i} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right). \quad (6)$$

Finally from Proposition 3.2, (3) and (6) we obtain

$$\begin{aligned} 1 = \mathbf{m}\left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) &\leq \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \leq \\ &\leq \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \leq 1 \end{aligned}$$

But $\mathbf{m}(0_{\mathcal{F}}((-\infty, t))) = 1$ for $t > 0$, hence

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}(0_{\mathcal{F}}((-\infty, t))).$$

“ $\Leftarrow$ ” Let $\mathbf{m}(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n((-\infty, t - \frac{1}{p}))) = \mathbf{m}(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n((-\infty, t - \frac{1}{p}))) = \mathbf{m}(0_{\mathcal{F}}((-\infty, t)))$ for every $t \in R$. We have to prove that a sequence $(x_n)_n$ of an IF -observables converges $\mathbf{m}$ -almost everywhere to 0, i.e.

$$\mathbf{m}\left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1,$$

what is equivalent with

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(R \setminus \left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 0. \quad (7)$$

Evidently

$$x_n \left(R \setminus \left(-\frac{1}{p}, \frac{1}{p} \right) \right) = x_n \left(\left(-\infty, -\frac{1}{p} \right] \right) \oplus x_n \left(\left[\frac{1}{p}, \infty \right) \right),$$

because $R \setminus \left(-\frac{1}{p}, \frac{1}{p} \right) = \left(-\infty, -\frac{1}{p} \right] \cup \left[\frac{1}{p}, \infty \right)$ and therefore

$$\bigvee_{n=k}^{\infty} x_n \left(R \setminus \left(-\frac{1}{p}, \frac{1}{p} \right) \right) \leq \bigvee_{n=k}^{\infty} x_n \left(\left(-\infty, -\frac{1}{p} \right] \right) \oplus \bigvee_{n=k}^{\infty} x_n \left(\left[\frac{1}{p}, \infty \right) \right).$$

Hence

$$\mathbf{m} \left(\bigvee_{n=k}^{\infty} x_n \left(R \setminus \left(-\frac{1}{p}, \frac{1}{p} \right) \right) \right) \leq \mathbf{m} \left(\bigvee_{n=k}^{\infty} x_n \left(\left(-\infty, -\frac{1}{p} \right] \right) \right) + \mathbf{m} \left(\bigvee_{n=k}^{\infty} x_n \left(\left[\frac{1}{p}, \infty \right) \right) \right). \quad (8)$$

But

$$\mathbf{m} \left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left(\left(-\infty, -\frac{1}{p} \right] \right) \right) = \mathbf{m}(\underline{x}((-\infty, 0))) = \mathbf{m}(0_{\mathcal{F}}((-\infty, 0))) = 0. \quad (9)$$

On the other hand

$$\mathbf{m} \left(\bigvee_{q=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} - \frac{1}{q} \right) \right) \right) = \mathbf{m} \left(\bar{x} \left(\left(-\infty, \frac{1}{p} \right) \right) \right) = \mathbf{m} \left(0_{\mathcal{F}} \left(\left(-\infty, \frac{1}{p} \right) \right) \right) = 1.$$

Since the last equality is equivalent with

$$\lim_{q \rightarrow \infty} \mathbf{m} \left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} - \frac{1}{q} \right) \right) \right) = 1,$$

to each $\varepsilon > 0$ there exists q such that

$$1 - \varepsilon < \mathbf{m} \left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} - \frac{1}{q} \right) \right) \right) \leq \mathbf{m} \left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} \right) \right) \right) \leq 1.$$

Since the relation holds for every $\varepsilon > 0$, then we have

$$\mathbf{m} \left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} \right) \right) \right) = 1.$$

Hence by this equality holding for every $p \in \mathbb{N}$ we obtain

$$\mathbf{m} \left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} \right) \right) \right) = 1. \quad (10)$$

Finally using (10) we have

$$\mathbf{m} \left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left(\left[\frac{1}{p}, \infty \right) \right) \right) = 1 - \mathbf{m} \left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, \frac{1}{p} \right) \right) \right) = 1 - 1 = 0. \quad (11)$$

By (8), (9) and (11) we obtain (7), i.e., a sequence $(x_n)_n$ of an IF -observables converges $\mathbf{m}$ -almost everywhere to 0. $\square$

In accordance to Proposition 4.1, we can extend the notion of $\mathbf{m}$ -almost everywhere convergence by the following way.

Definition 4.5. A sequence $(x_n)_n$ of IF-observables converges $\mathbf{m}$ -almost everywhere to an IF-observable x , if

$$\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}(x((-\infty, t))),$$

for every $t \in R$.

5 Conclusion

The paper is concerned in a modification of $\mathbf{m}$ -almost everywhere convergence with the help of $\limsup$ and $\liminf$ and the construction of translations formulas for these limits in an intuitionistic fuzzy case. We compare two kinds of definition of the $\mathbf{m}$ -almost everywhere convergence for intuitionistic fuzzy events and we show that they are equivalent.

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