Notes on Intuitionistic Fuzzy Sets
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k-Intuitionistic fuzzy structures

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Abstract: A more natural and necessary generalization of the intuitionistic fuzzy theory is developed and discussed in this paper. The generalization fits very nicely with almost all the intuitionistic fuzzy algebraic structures as well as with the intuitionistic fuzzy topological structures available in the literature. The higher dimensional intuitionistic fuzzy theory developed here helps us to define and discuss the concept of negation (complement) of a higher dimensional intuitionistic fuzzy set in a more natural way. In this paper we prove many theorems in the new context in both algebraic and topological points of view.
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1 Introduction

One cannot classify a student as intelligent or not, and a person as good or not; so an ambiguity arises while considering classes like the class of all intelligent students and class of all good people. Yet the fact remains that, such vaguely defined classes play a vital role in the society. Keeping this in mind, in 1965 Zadeh [16] introduced the concept of fuzzy sets, as functions from a set X to the closed interval $[0, 1]$ to study the uncertainties as a gradual membership of an object in a set.

In 1966, Goguen [10] extended the concept by defining fuzzy sets to be functions from a set into a lattice. Many others [7–9, 13–15] studied and developed the concept of fuzzy. In 1983, Atanassov [1] developed the concept of intuitionistic fuzzy subset, as an object having the form

$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$, where μ_A and ν_A are functions from the set X to $[0, 1]$ satisfying $\mu(x) + \nu(x) \leq 1$ for all $x \in X$ and defined some new operations over the intuitionistic fuzzy sets in [2].

The fuzzy theory helps us to rate a single property of an individual; for example the intelligence of a student, whereas the intuitionistic fuzzy theory helps us to discuss about the acceptance as well as the opposition of an individual; for example, how much the citizens of a country like and dislike their prime minister.

But in realistic situations the concept is much complicated. For example, while electing a minister, a citizen will have his own expectation; a voter may think the man who is going to be his minister should have some qualifications like education, easily approachable, good physic, capable of making own decisions with some gradation. Clearly it is not possible for one single person to posses all the qualifications. So, a voter has to convince himself, in certain qualifications of his minister; thus the matter of acceptance and opposition of qualification level, plays a vital role in voter's decision making, to cast his vote to a particular person.

Thus to discuss about the level of acceptance and the level of opposition, of a finite set of properties both fuzzy theory as well as the intuitionistic fuzzy theory are in some sense insufficient. To discuss similar problems, Atanassov defined and developed a theory of intuitionistic fuzzy sets of multi-dimension [3–6]. In [3], intuitionistic fuzzy multi-dimensional sets are defined as an object of the form

$$\langle x, \mu_A(x, z_1, z_2, \ldots, z_n), \nu_A(x, z_1, z_2, \ldots, z_n)\rangle$$

where μ_A and ν_A are functions from $E \times Z_1 \times Z_2 \cdots \times Z_n$ to I where $E, Z_1, Z_2, \ldots, Z_n$ are fixed sets.

In this paper, to study the level of acceptance and the level of opposition, of a finite set of properties, we develop a new structure called k-intuitionistic fuzzy structure and we discuss some theory on this new concept in the context of algebra as well as topology. The theory developed here shows that the concept of k-intuitionistic fuzzy theory nicely fits with both algebraic and topological setup. This can be further developed where ever fuzzy theory can be discussed.

In Section 2, we define a k-intuitionistic fuzzy subset and fix some notations; in Section 3, we develop our theory in context of algebra; in the Section 4, we develop our theory in the context of topology.

2 k-intuitionistic fuzzy structures

For any set X, a function $\mu : X \to [0, 1]$ is called a fuzzy subset of X. Throughout this paper, by I and I^X we denote the closed interval $[0, 1]$ and the set of all fuzzy subsets of X.

In [3], intuitionistic fuzzy multi-dimensional theory is studied using two functions μ_A and ν_A from $E \times Z_1 \times Z_2 \cdots \times Z_n$ to I. But in this paper, the theory is studied using two k-tuples $(\mu_1, \mu_2, \ldots, \mu_k)$ and $(\nu_1, \nu_2, \ldots, \nu_k)$ of fuzzy subsets of a set X. This way of seeing higher dimensional intuitionistic fuzzy theory helps us to define and discuss the concept of negation (complement) of a higher dimensional intuitionistic fuzzy set in a more natural way.

Definition 2.1. *Let X be a nonempty set and let k be a positive integer. Then a k-intuitionistic fuzzy subset of a set X is an ordered $2k$-tuple $(\mu_1, \mu_2, \ldots, \mu_k, \nu_1, \nu_2, \ldots, \nu_k)$ of functions from X to I satisfying*

$$\mu_i(x) + \nu_i(x) \leq 1 \text{ for all } \quad i = 1, 2, \ldots, k$$

and for all $x \in X$.

We abbreviate a k-intuitionistic fuzzy subset as k-ifs and we denote a k-ifs A as the ordered $2k$-tuple $(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_k}, \nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_k})$ throughout this paper.

Definition 2.2. *For any two k-intuitionistic fuzzy subsets A and B of a set X, we define*

- $A \subseteq B$ *if* $\mu_{A_i}(x) \leq \mu_{B_i}(x)$ *and* $\nu_{A_i}(x) \geq \nu_{B_i}(x)$, *for all* $x \in X$ *and for all* $i = 1, 2, \ldots, k$.
- $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$.
- $\bar{A}(x) = (\nu_{A_1}(x), \ldots, \nu_{A_k}(x), \mu_{A_1}(x), \ldots, \mu_{A_k}(x))$, *for all* $x \in X$.
- $(A \cap B)(x) = ((\mu_{A_1}(x) \wedge \mu_{B_1}(x)), \ldots, (\mu_{A_k}(x) \wedge \mu_{B_k}(x)),$
  $(\nu_{A_1}(x) \vee \nu_{B_1}(x)), \ldots, (\nu_{A_k}(x) \vee \nu_{B_k}(x)))$, *for all* $x \in X$.
- $(A \cup B)(x) = (\mu_{A_1}(x) \vee \mu_{B_1}(x)), \ldots, (\mu_{A_k}(x) \vee \mu_{B_k}(x)),$
  $(\nu_{A_1}(x) \wedge \nu_{B_1}(x)), \ldots, (\nu_{A_k}(x) \wedge \nu_{B_k}(x))$, *for all* $x \in X$.

Definition 2.3. *Let f be a function from X to Y and let*

$$A = (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_k}, \nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_k})$$

be a k-ifs in X. The image of A, written as $f(A)$ is a k-ifs in Y is given by,

$$f(A) = (\mu_{f(A)_1}, \mu_{f(A)_2}, \ldots, \mu_{f(A)_k}, \nu_{f(A)_1}, \nu_{f(A)_2}, \ldots, \nu_{f(A)_k})$$

where,

$$\mu_{f(A)_i}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_{A_i}(x)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)_i}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\nu_{A_i}(x)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.4. *Let f be a function from X to Y and let*

$$A = (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_k}, \nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_k})$$

be a k-ifs in Y. Then the inverse of A is written as $f^{-1}(A)$ is a k-ifs in X given by,

$$f^{-1}(A) = (\mu_{f^{-1}(A)_1}, \ldots, \mu_{f^{-1}(A)_k}, \nu_{f^{-1}(A)_1}, \ldots, \nu_{f^{-1}(A)_k})$$

where $\mu_{f^{-1}(A)_i}(x) = \mu_{A_i}(f(x))$ and $\nu_{f^{-1}(A)_i}(x) = \nu_{A_i}(f(x))$, $i = 1, 2, \ldots, k$ and for all $x \in X$.

3 k-Intuitionistic Fuzzy Algebraic Structures

For algebraic terminologies which are not defined explicitly in this paper we refer to [11].

Definition 3.1. *Let G be a group. A k-intuitionistic fuzzy subset A of G is said to be a k-intuitionistic fuzzy subgroup of G if it satisfies:*

i. $\mu_{A_i}(xy) \geq (\mu_{A_i}(x) \wedge \mu_{A_i}(y))$

ii. $\mu_{A_i}(x^{-1}) \geq \mu_{A_i}(x)$

iii. $\nu_{A_i}(xy) \leq (\nu_{A_i}(x) \vee \nu_{A_i}(y))$

iv. $\nu_{A_i}(x^{-1}) \leq \nu_{A_i}(x)$

for all $i = 1, 2, \ldots, k$ and for all $x, y \in G$.

The following theorems can be proved easily.

Theorem 3.2. *Let G be a group. Let A and B be any two k-intuitionistic fuzzy subgroups. Then, $A \cap B$ is a k-intuitionistic fuzzy subgroup.*

Theorem 3.3. *If A is a k-intuitionistic fuzzy subgroup of a group G. Then,*

i. $\mu_{A_i}(x^{-1}) = \mu_{A_i}(x)$

ii. $\nu_{A_i}(x^{-1}) = \nu_{A_i}(x)$

iii. $\mu_{A_i}(x) \leq \mu_{A_i}(e)$

iv. $\nu_{A_i}(x) \geq \nu_{A_i}(e)$

for all $i = 1, 2, \ldots, k$ and for all $x \in G$.

Theorem 3.4. *If A is a k-intuitionistic fuzzy subgroup of a group G. Then,*

i. $\mu_{A_i}(xy^{-1}) = \mu_{A_i}(e) \Rightarrow \mu_{A_i}(x) = \mu_{A_i}(y)$

ii. $\nu_{A_i}(xy^{-1}) = \nu_{A_i}(e) \Rightarrow \nu_{A_i}(x) = \nu_{A_i}(y)$

for all $i = 1, 2, \ldots, k$ and for all $x, y \in G$.

Theorem 3.5. *Let G be a group and A be a k-ifs of G. Then A is a k-intuitionistic fuzzy subgroup of a group G if and only if*

$$\mu_{A_i}(xy^{-1}) \geq \mu_{A_i}(x) \wedge \mu_{A_i}(y) \text{ and } \nu_{A_i}(xy^{-1}) \leq \nu_{A_i}(x) \vee \nu_{A_i}(y),$$

for all $i = 1, 2, \ldots, k$ and for all $x, y \in G$.

Theorem 3.6. *Let G and G' be two groups and $f : G \to G'$ be an onto homomorphism. Let A be a k-intuitionistic fuzzy subgroup of G. Then $f(A)$ is a k-intuitionistic fuzzy subgroup of G'.*

Proof. Let A be a k-intuitionistic fuzzy subgroup of G. We have to prove that $f(A)$ is a k-intuitionistic fuzzy subgroup of G'. If $u \in f^{-1}(x)$ and $v \in f^{-1}(y)$, then $f(u) = x$ and $f(v) = y$; as f is an onto homomorphism which implies that, $f(u)f(v) = f(uv) = xy$ and hence $uv \in f^{-1}(xy)$. Thus for every pair (u, v) where $u \in f^{-1}(x)$ and $v \in f^{-1}(y)$, we get an element $uv \in f^{-1}(xy)$. Since f is an onto homomorphism, the sets $f^{-1}(x)$ and $f^{-1}(y)$ are not empty and hence $f^{-1}(xy)$ is non empty. Now,

$$\begin{aligned}
\mu_{f(A)_i}(xy) &= \sup_{z\in f^{-1}(xy)} \{\mu_{A_i}(z)\} \\
&\geq \sup_{uv\in f^{-1}(xy)} \{\mu_{A_i}(uv)\} \\
&\geq \sup_{uv\in f^{-1}(xy)} \{(\mu_{A_i}(u) \wedge \mu_{A_i}(v))\} \\
&\quad (\text{ Since } \mu_{A_i}(uv) \geq \mu_{A_i}(u) \wedge \mu_{A_i}(v) \ ) \\
&\geq \sup_{u\in f^{-1}(x)} \{(\mu_{A_i}(u))\} \wedge \sup_{v\in f^{-1}(y)} \{(\mu_{A_i}(v))\} \\
&= \mu_{f(A)_i}(x) \wedge \mu_{f(A)_i}(y).
\end{aligned}$$

Hence (i) of Definition 3.1 follows. Similarly, we can prove the other conditions in Definition 3.1. □

Similarly, we can prove the following theorem.

Theorem 3.7. *Let G and G' be two groups and $f : G \to G'$ be an homomorphism. Let A' is a k-intuitionistic fuzzy group of G', then $f^{-1}(A')$ is a k-intuitionistic fuzzy subgroup of G.*

Definition 3.8. *Let A be a k-ifs of a set X and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, where $\alpha_i, \beta_i \in [0, 1]$, for all i. Then a (α, β) level subset of k-ifs A is defined as*

$$A_{(\alpha,\beta)} = \{x \in X/\mu_{A_i}(x) \geq \alpha_i \text{ and } \nu_{A_i}(x) \leq \beta_i, \text{ for all } i\}.$$

Theorem 3.9. *If A and B be two k-ifs, then the following properties hold:*

i. $A_{(\alpha,\beta)} \subseteq A_{(\alpha',\beta')}$ *if* $\alpha \geq \alpha'$ *and* $\beta \leq \beta'$.

ii. $A \subseteq B \Rightarrow A_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.

iii. $(A \cap B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)}$.

iv. $A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)} \subseteq (A \cup B)_{(\alpha,\beta)}$ *equality holds if* $\alpha_i + \beta_i = 1 \ \forall \ i$.

Theorem 3.10. *Let A be a k-intuitionistic fuzzy subset of a group G. Then A is a k-intuitionistic fuzzy subgroup of G if and only if all (α, β) level subsets of A, are subgroups of G.*

Proof. Let $A = (\mu_{A_1}, \ldots, \mu_{A_k}, \nu_{A_1}, \ldots, \nu_{A_k})$ be a k-intuitionistic fuzzy subgroup of G and let $A_{(\alpha,\beta)}$ be a (α, β) level subset of A. We claim that $A_{(\alpha,\beta)}$ is a subgroup of G. Let $x, y \in A_{(\alpha,\beta)}$. Then, we have

$$\mu_{A_i}(x) \geq \alpha_i, \ \nu_{A_i}(x) \leq \beta_i \ \text{ and } \ \mu_{A_i}(y) \geq \alpha_i, \ \nu_{A_i}(y) \leq \beta_i,$$

for all $i = 1, 2, \ldots, k$. Now,

$$\mu_{A_i}(xy^{-1}) \geq \mu_{A_i}(x) \wedge \mu_{A_i}(y^{-1}) = \mu_{A_i}(x) \wedge \mu_{A_i}(y) \geq \alpha_i \wedge \alpha_i = \alpha_i,$$

for all i. Similarly, we can prove that, $\nu_{A_i}(xy^{-1}) \leq \beta_i$, for all i. Thus, $xy^{-1} \in A_{(\alpha,\beta)}$ and hence the claim follows.

Conversely, assume that all (α, β) level subsets of A are subgroups of G. We claim that A is k-intuitionistic fuzzy subgroup of G. Let $x, y \in G$. Now since $A = (\mu_{A_1}, \ldots, \mu_{A_k}, \nu_{A_1}, \ldots, \nu_{A_k})$ is an k-ifs, there exists some $a_i, b_i, c_i, d_i \in [0, 1]$ such that $\mu_{A_i}(x) = a_i$, $\nu_{A_i}(x) = b_i$, $\mu_{A_i}(y) = c_i$ and $\nu_{A_i}(y) = d_i$, for all i.

Let $\alpha_i = a_i \wedge c_i$ and $\beta_i = b_i \vee d_i$ for all i. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ then, $x, y \in A_{(\alpha,\beta)}$. Now since $A_{(\alpha,\beta)}$ is a subgroup of G, we have $xy \in A_{(\alpha,\beta)}$.

Therefore, $\mu_{A_i}(xy) \geq \alpha_i$ and $\nu_{A_i}(xy) \leq \beta_i$, for all i and hence $\mu_{A_i}(xy) \geq \mu_{A_i}(x) \wedge \mu_{A_i}(y)$ and $\nu_{A_i}(xy) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$ for all i. Thus (i) and (iii) of Definition 3.1 follows. Similarly, we can prove (ii) and (iv) of Definition 3.1. Thus A is k-intuitionistic fuzzy subgroup of G. □

Definition 3.11. *A k-intuitionistic fuzzy subgroup A, of a group G, is said to be a k-intuitionistic fuzzy normal subgroup of G, if*

$$\mu_{A_i}(xyx^{-1}) \geq \mu_{A_i}(y) \text{ and } \nu_{A_i}(xyx^{-1}) \leq \nu_{A_i}(y)$$

for all $i = 1, 2, \ldots, k$ and for all $x, y \in G$.

Theorem 3.12. *Let A be a k-intuitionistic fuzzy subgroup of a group G. Then the following conditions are equivalent:*

i. *A is k-intuitionistic fuzzy normal.*

ii. *$\mu_{A_i}(xyx^{-1}) = \mu_{A_i}(y)$ and $\nu_{A_i}(xyx^{-1}) = \nu_{A_i}(y)$ for all $x, y \in G$ and for all $i = 1, 2, \ldots, k$.*

iii. *$\mu_{A_i}(xy) = \mu_{A_i}(yx)$ and $\nu_{A_i}(xy) = \nu_{A_i}(yx)$ for all $x, y \in G$ and for all $i = 1, 2, \ldots, k$.*

iv *Each (α, β) level subset $A_{(\alpha,\beta)}$ of A is a normal subgroup of G.*

Theorem 3.13. *Let A and A' be k-intuitionistic fuzzy normal subgroups of G and G' respectively. Let $f : G \to G'$ be a homomorphism. Then the following statements are true.*

i. *$f(A)$ is a k-intuitionistic fuzzy normal subgroup of G' if f is onto.*

ii. *$f^{-1}(A')$ is a k-intuitionistic fuzzy normal subgroup of G.*

Proof. Let A and A' be k-intuitionistic fuzzy normal subgroups of G and G' and let $f : G \to G'$ be an onto homomorphism. Then by Theorem 3.6 it follows that $f(A)$ is a k-intuitionistic fuzzy subgroup of G'. So it is enough to prove that, $\mu_{f(A)_i}(xy) = \mu_{f(A)_i}(yx)$ and $\nu_{f(A)_i}(xy) = \nu_{f(A)_i}(yx)$, for all i and for all $x, y \in G'$.

Let $x, y \in G'$, then there exist u and v in G such that, $f(u) = x$ and $f(v) = y$. Since A is an k-ifs of G, there exists $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ in $[0, 1]$ such that $\mu_{A_i}(u) = a_i$, $\nu_{A_i}(u) = b_i$,

$\mu_{A_i}(v) = c_i$ and $\mu_{A_i}(v) = d_i$. Let $\alpha_i = a_i \wedge c_i$ and $\beta_i = b_i \vee d_i$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ then, $u, v \in A_{(\alpha,\beta)}$. This implies that $f(u), f(v) \in f(A_{(\alpha,\beta)})$. That is $x, y \in f(A_{(\alpha,\beta)})$. Now, since A is a k-intuitionistic fuzzy normal subgroup of G, by Theorem 3.12 it follows that, every (α, β) level subset $A_{(\alpha,\beta)}$ of A is a normal subgroup of G. Also, we know that the homomorphic image of a normal subgroup is normal subgroup. This implies that, $f(A_{(\alpha,\beta)})$ is a normal subgroup of G'. Thus, we have $xy = yx$. Since the above arguement holds for all $x, y \in G'$, it follows that, $\mu_{f(A)_i}(xy) = \mu_{f(A)_i}(yx)$ and $\nu_{f(A)_i}(xy) = \nu_{f(A)_i}(yx)$, for all i and for all $x, y \in G'$.

Now we claim that, $f^{-1}(A')$ is a k-intuitionistic fuzzy normal subgroup of G. By Theorem 3.7 it follows that $f^{-1}(A')$ is a k-intuitionistic fuzzy subgroup of G. So, it is enough to prove that $f^{-1}(A')$ k-intuitionistic fuzzy normal in G. Let $x, y \in G$. Then, for all i and for all $x, y \in G$,

$$\begin{aligned}
\mu_{f^{-1}(A)'_i}(xy) &= \mu_{A'_i}(f(xy)) \\
&= \mu_{A'_i}(f(x)f(y)) \qquad \text{(since f is a group homomorphism)} \\
&= \mu_{A'_i}(f(y)f(x)) \quad \text{(since } A' \text{ is a } k\text{-intuitionistic fuzzy} \\
&\qquad\qquad\qquad\qquad\qquad \text{normal subgroup of } G') \\
&= \mu_{A'_i}(f(yx)) \\
&= \mu_{f^{-1}(A)'_i}(yx).
\end{aligned}$$

Similarly $\nu_{f^{-1}(A)'_i}(xy) = \nu_{f^{-1}(A)'_i}(yx)$ for all i and for all $x, y \in G$. □

Definition 3.14. *A k-intuitionistic fuzzy subgroup A of a group G is said to be a k-intuitionistic fuzzy characteristic subgroup of G, if it satisfies the following conditions:*

i. $\mu_{A_i}(g) = \mu_{A_i}(f(g))$

ii. $\nu_{A_i}(g) = \nu_{A_i}(f(g))$,

for all $i = 1, 2, \ldots, k,\ g \in G$ *and* $f \in Aut(G)$.

Theorem 3.15. *Let A be a k-intuitionistic fuzzy subgroup of a group G. Then the following statements are equivalent.*

i. A is a k-intuitionistic fuzzy characteristic subgroup of G.

ii. Each (α, β) level subset $A_{(\alpha,\beta)}$ of A is a characteristic subgroup of G.

Proof. Assume that A is a k-intuitionistic fuzzy characteristic subgroup of G. Let $A_{(\alpha,\beta)}$ be a (α, β) level subset of A. We claim that $A_{(\alpha,\beta)}$ is a characteristic subgroup of G. That is to prove that $f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$ for all $f \in Aut(G)$. Let $f \in Aut(G)$ and $y \in f(A_{(\alpha,\beta)})$. Then there exists $x \in A_{(\alpha,\beta)}$ such that $f(x) = y$. Since A is a k-intuitionistic fuzzy characteristic subgroup of G, we have,

$$\mu_{A_i}(f(g)) = \mu_{A_i}(g) \text{ and } \nu_{A_i}(g) = \nu_{A_i}(f(g)),$$

for all $i = 1, 2, \ldots, k,\ g \in G$ and $f \in Aut(G)$. This implies that,

$$\mu_{A_i}(y) = \mu_{A_i}(f(x)) = \mu_{A_i}(x) \geq \alpha_i,$$

for all i. Similarly, we can prove that $\nu_{A_i}(y) \leq \beta_i$, for all i and hence it follows that, $y \in A_{(\alpha,\beta)}$. Thus, $f(A_{(\alpha,\beta)}) \subseteq A_{(\alpha,\beta)}$. Similarly, we can prove the reverse inequality.

Conversely, assume that each (α, β) level subset $A_{(\alpha,\beta)}$ of A is a characteristic subgroup of G. We claim that A is a k-intuitionistic fuzzy characteristic subgroup of G.

Let $g \in G$ and $f \in Aut(G)$.

Since A is a k-ifs of G, there exist $\alpha_i, \beta_i \in [0, 1]$ such that, $\mu_{A_i}(g) = \alpha_i$ and $\nu_{A_i}(g) = \beta_i$, for all i. This implies that $g \in A_{(\alpha,\beta)}$, where $(\alpha, \beta) = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k)$. Now, since $g \in A_{(\alpha,\beta)}$ we have $f(g) \in f(A_{(\alpha,\beta)})$. But by our assumption, we have $f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$, for all (α, β) level subsets of A. Hence it follows that, $f(g) \in A_{(\alpha,\beta)}$. This implies that, $\mu_{A_i}(f(g)) \geq \alpha_i$ and $\nu_{A_i}(f(g)) \leq \beta_i$, for all i.

Now, we claim that, $\mu_{A_i}(f(g)) = \alpha_i$ and $\nu_{A_i}(f(g)) = \beta_i$, for all i. Suppose $\mu_{A_j}(f(g)) > \alpha_j$ and $\nu_{A_j}(f(g)) < \beta_j$, for some $1 \leq j \leq k$. Then there exist δ_j and θ_j such that $\delta_j > \alpha_j$ and $\theta_j < \beta_j$ with $\mu_{A_j}(f(g)) = \delta_j$ and $\nu_{A_j}(f(g)) = \theta_j$. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$, where $\delta_i = 0 = \theta_i$ if $i \neq j$. Then, clearly $f(g) \in A_{(\delta,\theta)} = f(A_{(\delta,\theta)})$ and thus $g \in A_{(\delta,\theta)}$. This implies that, $\mu_{A_i}(g) \geq \delta_i$ and $\nu_{A_i}(g) \leq \theta_i$, for all i. But this is not possible since,

$$\mu_{A_j}(g) = \alpha_j < \delta_j \text{ and } \nu_{A_j}(g) = \beta_j > \theta_j.$$

Thus our assumption $\mu_{A_j}(f(g)) > \alpha_j$ and $\nu_{A_j}(f(g)) < \beta_j$, for some $1 \leq j \leq k$, is wrong. Hence, $\mu_{A_i}(f(g)) = \alpha_i$ and $\nu_{A_i}(f(g)) = \beta_i$, for all i. □

Theorem 3.16. *Every k-intuitionistic fuzzy characteristic subgroup of a group G is a k-intuitionistic fuzzy normal subgroup of G.*

Proof. Assume A be any k-intuitionistic fuzzy characteristic subgroup of G. Let $f : G \to G$ be a function defined by $f(x) = yxy^{-1}$, then $f \in Aut(G)$. Now, let $x, y \in G$, then

$$\mu_{A_i}(xy) = \mu_{A_i}(f(xy)) = \mu_{A_i}(y(xy)y^{-1}) = \mu_{A_i}(yx(yy^{-1})) = \mu_{A_i}(yx),$$

for all i and for all $x, y \in G$. □

4 k-Intuitionistic fuzzy topological structures

Let $(\underline{0}, \underline{0}, \ldots, \underline{0}, \underline{1}, \underline{1}, \ldots, \underline{1})$ and $(\underline{1}, \underline{1}, \ldots, \underline{1}, \underline{0}, \underline{0}, \ldots, \underline{0})$ be k-intuitionistic fuzzy sets, where $\underline{1}$ and $\underline{0}$ are the constant maps defined by $\underline{1}(x) = 1$, for all $x \in X$ and $\underline{0}(x) = 0$, for all $x \in X$. We denote the k-ifs $(\underline{0}, \underline{0}, \ldots, \underline{0}, \underline{1}, \underline{1}, \ldots, \underline{1})$ by $\tilde{0}$ and the k-ifs $(\underline{1}, \underline{1}, \ldots, \underline{1}, \underline{0}, \underline{0}, \ldots, \underline{0})$ by $\tilde{1}$. The k-ifs $\tilde{0}$ and $\tilde{1}$ of a set X are also denoted by $\emptyset$ and X.

For topological terminologies which are not defined explicitly in this paper we refer to [12].

Definition 4.1. *A k-intuitionistic fuzzy topology on a non-empty set X, is a family $\mathcal{T}$ of k-intuitionistic fuzzy subsets of X which satisfy the following conditions :*

i. $\tilde{0}, \tilde{1} \in \mathcal{T}$.

ii. $G_1 \cap G_2 \in \mathcal{T}$, *for all* $G_1, G_2 \in \mathcal{T}$.

iii. $\cup G_\lambda \in \mathcal{T}$, *for any arbitrary collection,* $\{G_\lambda / G_\lambda \in \mathcal{T}\}_{\lambda\in\Lambda}$.

A set for which a k-intuitionistic fuzzy topology is specified, is called a k-intuitionistic fuzzy topological space. A k-intuitionistic fuzzy topological space is denoted by the pair $(X, \mathcal{T})$.

Elements of $\mathcal{T}$ are called k-intuitionistic fuzzy open sets. A k-intuitionistic fuzzy subset is said to be closed if its complement is open. We abbreviate a k-intuitionistic fuzzy closed set as k-ifcs and k-intuitionistic fuzzy open set as k-ifos.

Theorem 4.2. *Let $\{\mathcal{T}_\lambda / \lambda \in \Lambda\}$ be a family of k-intuitionistic fuzzy topologies on X. Then $\cap\mathcal{T}_\lambda$ is also a k-intuitionistic fuzzy topology on X. Furthermore, $\cap\mathcal{T}_\lambda$ is the coarsest k-intuitionistic fuzzy topology on X containing all $\mathcal{T}_\lambda$.*

Definition 4.3. *Let $(X, \mathcal{T})$ be a k-intuitionistic fuzzy topological space and let $A = (\mu_{A_1}, \ldots, \mu_{A_k}, \nu_{A_1}, \ldots, \nu_{A_k})$ be a k-ifs in X. Then k-intuitionistic fuzzy interior and k-intuitionistic fuzzy closure of A are defined by,*

$$cl(A) = \cap\{F/F \text{ is an k-ifcs in X and } A \subseteq F\}$$

$$int(A) = \cup\{G/G \text{ is an k-ifos in X and } G \subseteq A\}$$

Note that $cl(A)$ is a k-ifcs and $int(A)$ is a k-ifos in X. Further,

- A is a k-ifcs in X if and only if $cl(A) = A$.
- A is a k-ifos in X if and only if $int(A) = A$.

Definition 4.4. *Let $(X, \mathcal{T}), (Y, \sigma)$ be two k-intuitionistic fuzzy topological spaces and let $f : X \to Y$ be a function. Then f is said to be k-intuitionistic fuzzy continuous if inverse image of each k-ifs in σ is a k-ifs in $\mathcal{T}$.*

Definition 4.5. *Let $(X, \mathcal{T})$ and (Y, σ) be two k-intuitionistic fuzzy topological spaces and let $f : X \to Y$ be a function. Then f is said to be k-intuitionistic fuzzy open if image of each k-ifs in $\mathcal{T}$ is a k-ifs in σ.*

Theorem 4.6. *Let $(X, \mathcal{T})$ and (Y, σ) be two k-intuitionistic fuzzy topological spaces. Then a function $f : (X, \mathcal{T}) \to (Y, \sigma)$ is k-intuitionistic fuzzy continuous if and only if the inverse image of each k-ifcs in σ is a k-ifcs in $\mathcal{T}$.*

Theorem 4.7. *Let $(X, \mathcal{T})$ and (Y, σ) be two k-intuitionistic fuzzy topological spaces and let f be a function from $(X, \mathcal{T})$ to (Y, σ). Then following statements are equivalent:*

i. $f : (X, \mathcal{T}) \to (Y, \sigma)$ *is fuzzy continuous.*

ii. $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ *for each k-ifs B in Y.*

iii. $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ *for each k-ifs B in Y.*

Proof. First let us assume that $f : (X, \mathcal{T}) \to (Y, \sigma)$ be k-intuitionistic fuzzy continuous. We claim that, $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ for each k-ifs B in Y.

Let $B = (\mu_{B_1}, \mu_{B_2}, \ldots, \mu_{B_k}, \nu_{B_1}, \nu_{B_2}, \ldots, \nu_{B_k})$, be a k-ifs of Y. Let

$$\{G_\lambda = (\mu_{G_{\lambda_1}}, \mu_{G_{\lambda_2}}, \ldots, \mu_{G_{\lambda_k}}, \nu_{G_{\lambda_1}}, \nu_{G_{\lambda_2}}, \ldots, \nu_{G_{\lambda_k}}) /\ \lambda \in \Lambda\},$$

be the collection of all k-intuitionistic fuzzy open sets contained in B, then by the definition of k-intuitionistic fuzzy interior of a k-ifs, we have

$$int(B) = \{(\vee\mu_{G_{\lambda_1}}, \ldots, \vee\mu_{G_{\lambda_k}}, \wedge\nu_{G_{\lambda_1}}, \ldots, \wedge\nu_{G_{\lambda_k}})/\lambda \in \Lambda\}.$$

Now,

$$\begin{aligned}
f^{-1}(int(B))(x) &= f^{-1}\big(\vee\, \mu_{G_{\lambda_1}}, \ldots, \vee\mu_{G_{\lambda_k}}, \wedge\nu_{G_{\lambda_1}}, \ldots, \wedge\nu_{G_{\lambda_k}}\big)(x) \\
&= \big((\vee\mu_{G_{\lambda 1}})(f(x)), \ldots, (\vee\mu_{G_{\lambda k}})(f(x)), \\
&\qquad (\wedge\nu_{G_{\lambda 1}})(f(x)), \ldots, (\wedge\nu_{G_{\lambda k}})(f(x))\big) \\
&= \big(\vee\,(\mu_{G_{\lambda 1}}(f(x))), \ldots, \vee(\mu_{G_{\lambda k}}(f(x))), \\
&\qquad \wedge(\nu_{G_{\lambda 1}}(f(x))), \ldots, \wedge(\nu_{G_{\lambda k}}(f(x)))\big) \\
&\quad \big(\vee\,(\mu_{f^{-1}(G_\lambda)_1})(x), \ldots, \vee(\mu_{f^{-1}(G_\lambda)_k})(x), \\
&\qquad \wedge(\nu_{f^{-1}(G_\lambda)_1})(x), \ldots, \wedge(\nu_{f^{-1}(G_\lambda)_1})(x)\big) \\
&= \vee f^{-1}(G_\lambda)(x).
\end{aligned}$$

Now, since $G_\lambda \subseteq B$ for all $\lambda \in \Lambda$, we have $f^{-1}(G_\lambda) \subseteq f^{-1}(B)$, for all $\lambda \in \Lambda$. By our assumption, f is a k-intuitionistic fuzzy continuous function and since G_λ is a k-ifos for all $\lambda \in \Lambda$, it follows that, $f^{-1}(G_\lambda)$ is k-ifos for all $\lambda \in \Lambda$. Therefore, $f^{-1}(G_\lambda)$ is a k-ifos contained in $f^{-1}(B)$ for all $\lambda \in \Lambda$. This implies that, $\vee f^{-1}(G_\lambda)$ is a k-ifos contained in $f^{-1}(B)$. Thus we have, $\vee f^{-1}(G_\lambda) \subseteq int(f^{-1}(B))$. But since $f^{-1}(int(B)) = \vee f^{-1}(G_\lambda)$, it follows that, $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$.

Now, let us assume that $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$, for each k-ifs B in Y. We claim that f is k-intuitionistic fuzzy continuous. Let $B \in \sigma$, then we have to prove that $f^{-1}(B) \in \mathcal{T}$. That is, we have to prove that, $int(f^{-1}(B)) = f^{-1}(B)$. Now since $B \in \sigma$ we have, $int(B) = B$ and hence $f^{-1}(int(B)) = f^{-1}(B)$. But by our assumption, it follows that, $f^{-1}(B) \subseteq int(f^{-1}(B))$. But we know that, for any k-ifs, its k-intuitionistic fuzzy interior is contained in itself. Thus, $f^{-1}(B) = int(f^{-1}(B))$.

Now, assume that f is a k-intuitionistic fuzzy continuous function from $(X, \mathcal{T})$ to (Y, σ). We claim that $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, for each k-ifs B in Y. Let

$$B = (\mu_{B_1}, \mu_{B_2}, \ldots, \mu_{B_k}, \nu_{B_1}, \nu_{B_2}, \ldots, \nu_{B_k}),$$

be a k-ifs in Y. Let

$$\{F_\lambda = (\mu_{F_{\lambda_1}}, \mu_{F_{\lambda_2}}, \ldots, \mu_{F_{\lambda_k}}, \nu_{F_{\lambda_1}}, \nu_{F_{\lambda_2}}, \ldots, \nu_{F_{\lambda_k}}) /\ \lambda \in \Lambda\},$$

be the collection of all k-intuitionistic fuzzy closed sets containing in B, then by the definition of k-intuitionistic fuzzy closure of a k-ifs, we have

$$cl(B) = \{(\wedge\mu_{F_{\lambda_1}}, \ldots, \wedge\mu_{F_{\lambda_k}}, \vee\nu_{F_{\lambda_1}}, \ldots, \vee\nu_{F_{\lambda_1}})/\lambda \in \Lambda\}.$$

Consider,

$$\begin{aligned}
f^{-1}(cl(B))(x) &= f^{-1}\big(\wedge \mu_{F_{\lambda_1}}, \ldots, \wedge\mu_{F_{\lambda_k}}, \vee\nu_{F_{\lambda_1}}, \ldots, \vee\nu_{F_{\lambda_1}}\big)(x) \\
&= \big((\wedge\mu_{F_{\lambda 1}})(f(x)), \ldots, (\wedge\mu_{F_{\lambda k}})(f(x)), \\
&\qquad (\vee\nu_{F_{\lambda 1}})(f(x)), \ldots, (\vee\nu_{F_{\lambda k}})(f(x))\big) \\
&= \big(\wedge(\mu_{F_{\lambda_1}}(f(x))), \ldots, \wedge(\mu_{F_{\lambda_k}}(f(x))), \\
&\qquad \vee(\nu_{F_{\lambda 1}}(f(x))), \ldots, \vee(\nu_{F_{\lambda k}}(f(x)))\big) \\
&\quad \big(\wedge(\mu_{f^{-1}(F_\lambda)_1})(x), \ldots, \wedge(\mu_{f^{-1}(F_\lambda)_k})(x), \\
&\qquad \vee(\nu_{f^{-1}(F_\lambda)_1})(x), \ldots, \vee(\nu_{f^{-1}(F_\lambda)_1})(x)\big) \\
&= \wedge f^{-1}(F_\lambda)(x).
\end{aligned}$$

This implies that, $f^{-1}(cl(B)) = \wedge f^{-1}(F_\lambda)$. Now, since $B \subseteq F_\lambda$ for all $\lambda \in \Lambda$, we have $f^{-1}(B) \subseteq f^{-1}(F_\lambda)$, for all $\lambda \in \Lambda$. By our assumption, f is a k-intuitionistic fuzzy continuous function and since F_λ is a k-ifcs for all $\lambda \in \Lambda$, it follows that, $f^{-1}(F_\lambda)$ is k-ifcs for all $\lambda \in \Lambda$. Thus, $f^{-1}(F_\lambda)$ is a k-ifcs containing $f^{-1}(B)$ for all $\lambda \in \Lambda$. This implies that, $\wedge f^{-1}(F_\lambda)$ is a k-ifcs containing $f^{-1}(B)$. Thus we have, $cl(f^{-1}(B)) \subseteq \wedge f^{-1}(G_\lambda)$. But since $f^{-1}(cl(B)) = \wedge f^{-1}(F_\lambda)$, it follows that, $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Now assume that $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for each k-ifs B in Y. We claim that f is k-intuitionistic fuzzy continuous, that is, we have to prove that the inverse image of each k-ifcs in Y is a k-ifcs in X. Let B be a k-ifcs of X, then we have to prove that $f^{-1}(B)$ is a k-ifcs of Y. Since B is k-ifcs, we have $B = cl(B)$ and thus it follows that, $f^{-1}(B) = f^{-1}(cl(B))$. But by our assumption, $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. This implies that, $cl(f^{-1}(B)) \subseteq f^{-1}(B)$. The reverse inequality follows obviously. Thus, $f^{-1}(B)$ is a k-intuitionistic fuzzy closed set in X. □

Definition 4.8. *Let $\mathscr{A}$ be a collection of k-intuitionistic fuzzy subsets of a set X. Then $\mathscr{A}$ is called a cover for a k-ifs B of X if*

$$B \subset \cup\{A/A \in \mathscr{A}\}.$$

Correspondingly, $\mathscr{A}$ is called an open cover if each member of $\mathscr{A}$ is a k-intuitionistic fuzzy open set. A subcover is a subfamily of $\mathscr{A}$ which is also a cover for B.

Definition 4.9. *Let E be a k-ifs of a k-intuitionistic fuzzy topological space $(X, \mathcal{T})$. Then E is said to be k-intuitionistic fuzzy compact if every open cover of E has a finite subcover.*

Definition 4.10. *A family $\mathscr{A}$ of k-ifs of X is said to have the finite intersection property if for every finite subfamily $\{A_1, A_2, \ldots, A_n\}$ of $\mathscr{A}$, the intersection $\bigcap_{i=1}^{n} A_n$ is non-empty.*

Theorem 4.11. *Let $\mathscr{A}$ be a collection of k-ifs in a k-intuitionistic fuzzy topological space $(X, \mathcal{T})$ and let $\mathscr{C} = \left\{\overline{A}/A \in \mathscr{A}\right\}$ be the collection of their complements, then the following holds.*

i. $\mathscr{A}$ is a collection of k-intuitionistic fuzzy open sets if and only if $\mathscr{C}$ is a collection of k-intuitionistic fuzzy closed sets.

ii. The collection $\mathscr{A}$ covers X if and only if the intersection $\underset{C\in\mathscr{C}}{\cap} C$ of all elements of $\mathscr{C}$ is empty.

iii. The finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathscr{A}$ covers X if and only if the intersection of the corresponding elements $C_\lambda = \overline{A_\lambda}$ of $\mathscr{C}$ is empty.

Proof. (i) follows trivially from the definition of k-ifos and k-ifcs. To prove (ii), first let us assume that, the collection $\mathscr{A}$ covers X, then we have, $X \subseteq \underset{\lambda\in\Lambda}{\cup} \{A_\lambda/A_\lambda \in \mathscr{A}\}$. That is,

$$\tilde{1} \subseteq \underset{\lambda\in\Lambda}{\cup} \{(\mu_{\lambda_1}, \ldots, \mu_{\lambda_k}, \nu_{\lambda_1}, \ldots, \nu_{\lambda_k})/A_\lambda \in \mathscr{A}\}.$$

This implies that,

$$(\underline{1}, \ldots, \underline{1}, \underline{0}, \ldots, \underline{0}) \subseteq \underset{\lambda\in\Lambda}{\cup} \{(\mu_{\lambda_1}, \ldots, \mu_{\lambda_k}, \nu_{\lambda_1}, \ldots, \nu_{\lambda_k})/A_\lambda \in \mathscr{A}\}.$$

This implies that,

$$(\underline{1}, \underline{1}, \ldots, \underline{1}, \underline{0}, \underline{0}, \ldots, \underline{0}) \subseteq (\vee\mu_{\lambda_1}, \ldots, \vee\mu_{\lambda_k}, \wedge\nu_{\lambda_1}, \ldots, \wedge\nu_{\lambda_k}).$$

Thus,

$$\overline{(\vee\mu_{\lambda_1}, \ldots, \vee\mu_{\lambda_k}, \wedge\nu_{\lambda_1}, \ldots, \wedge\nu_{\lambda_l})} \subseteq \overline{(\underline{1}, \underline{1}, \ldots, \underline{1}, \underline{0}, \underline{0}, \ldots, \underline{0})}.$$

This implies that,

$$(\wedge\nu_{\lambda_1}, \ldots, \wedge\nu_{\lambda_l}, \vee\mu_{\lambda_1}, \ldots, \vee\mu_{\lambda_k}) \subseteq (\underline{0}, \underline{0}, \ldots, \underline{0}, \underline{1}, \underline{1}, \ldots, \underline{1}) = \tilde{0}.$$

Thus, $(\underset{\lambda\in\Lambda}{\cap} \{\overline{A_\lambda}/A_\lambda \in \mathscr{A}\}) \subseteq \tilde{0}$, and hence $(\underset{\lambda\in\Lambda}{\cap} \{\overline{A_\lambda}/A_\lambda \in \mathscr{A}\}) = \emptyset$. Similarly, we can prove the converse.

The proof of (iii) is analogous to the proof of (ii). □

Theorem 4.12. *Let $(X, \mathcal{T})$ be a k-intuitionistic fuzzy topological space. Then X is k-intuitionistic fuzzy compact if and only if for every collection $\mathscr{C}$ of k-intuitionistic fuzzy closed sets in X having the finite intersection property, the intersection $\cap_{C\in\mathscr{C}} C$ of all elements of $\mathscr{C}$ is non-empty.*

Proof. If $\mathscr{A}$ is a family of k-ifs in a k-intuitionistic fuzzy topological space $(X, \mathcal{T})$. Let $\mathscr{C} = \{\bar{A}/A \in \mathscr{A}\}$ be the collection of their complements, then by Theorem 4.11 the following holds:

i. $\mathscr{A}$ is a collection of k-intuitionistic fuzzy open sets if and only if $\mathscr{C}$ is a collection of k-intuitionistic fuzzy closed sets.

ii. The collection $\mathscr{A}$ covers X if and only if the intersection $\cap_{C\in\mathscr{C}} C$ of all the elements of $\mathscr{C}$ is empty.

iii. The finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of $\mathscr{A}$ covers X if and only if the corresponding elements $C_i = \bar{A}_i$ of $\mathscr{C}$ is empty.

Now, the statement X is k-intuitionistic fuzzy compact is equivalent to: "Given any collection $\mathscr{A}$ of k-intuitionistic open sets of X, if A covers X, then some finite subcollection of $\mathscr{A}$ covers X." This statement is equivalent to the contrapositive which is the following: "Given any collection $\mathscr{A}$ of k-intuitionistic fuzzy open sets if no finite subcollection of $\mathscr{A}$ covers X. Then, $\mathscr{A}$ does not cover X."

Let $\mathscr{C}$ be the collection $\{\bar{A}/A \in \mathscr{A}\}$, then by Conditions (i) and (iii), we see that the above statement is in turn is equivalent to the following: "Given any collection $\mathscr{C}$ of k-intuitionistic fuzzy closed sets, if every finite intersection of elements of $\mathscr{C}$ is non-empty, then the intersection of all elements of $\mathscr{C}$ is non-empty." □

Theorem 4.13. *Let $(X, \mathcal{T})$ and (Y, σ) be two k-intuitionistic fuzzy topological spaces and let f be a continuous function X onto Y. If X is k-intuitionistic fuzzy compact then Y is k-intuitionistic fuzzy compact.*

Proof. Let $\mathscr{B}$ be an open cover of Y. Then, for any $x \in X$ We have,

$$\begin{aligned} \mu_{(\cup_{B\in\mathscr{B}} f^{-1}(B))_i}(x) &= \sup_{B\in\mathscr{B}}\{\mu_{f^{-1}(B)_i}(x)\} \text{ for all } \ i = 1, 2, \ldots, k \\ &= \sup_{B\in\mathscr{B}}\{\mu_{B_i}(f(x))\} \text{ for all } \ i = 1, 2, \ldots, k \\ &= 1 \text{ for all } \ i = 1, 2, \ldots, k. \end{aligned}$$

And for any $x \in X$,

$$\begin{aligned} \nu_{(\cup_{B\in\mathscr{B}} f^{-1}(B))_i}(x) &= \inf_{B\in\mathscr{B}}\{\nu_{f^{-1}(B)_i}(x)\} \text{ for all } \ i = 1, 2, \ldots, k \\ &= \inf_{B\in\mathscr{B}}\{\nu_{B_i}(f(x))\} \text{ for all } \ i = 1, 2, \ldots, k \\ &= 0 \text{ for all } \ i = 1, 2, \ldots, k. \end{aligned}$$

Note that, as f is k-intuitionistic fuzzy continuous $f^{-1}(B)$ is a k-ifos for all $B \in \mathscr{B}$. Thus family of all k-intuitionistic fuzzy sets of the form $f^{-1}(B)$, for B in $\mathscr{B}$, is an open cover for X. Since X is k-intuitonistic fuzzy compact, the corresponding collection has a finite subcover. Let

$$\{f^{-1}(B_1), \ldots, f^{-1}(B_n)/B_k \in \mathscr{B}, \text{ for all } k = 1, 2, \ldots, n\}$$

be the finite subcollection covering X. Then since f is onto, it can be easily seen that $f(f^{-1}(B)) = B$, for any fuzzy set B in Y. Thus, the finite subcollection $\{B_1, B_2, \ldots, B_n/B_k \in \mathscr{B}$, for all $k = 1, 2, \ldots, n\}$, is a finite subfamily of $\mathscr{B}$, which covers Y. □

5 Conclusion

In this paper, we discussed a more natural and necessary generalization of the intuitionistic fuzzy theory. We proved that this theory fits nicely with almost all algebraic and topological structures and the theory is consistent with the existing intuitionistic fuzzy theory. This can be further developed wherever fuzzy theory can be discussed.

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