**Notes on Intuitionistic Fuzzy Sets**
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# Related fixed point theorems in intuitionistic fuzzy metric spaces satisfying an implicit relation

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**Abstract:** In this paper, we introduce a new class of implicit relation to present an extended version of a fixed point theorem of Popa [23] in the framework of intuitionistic fuzzy metric space.

**Keywords:** Common fixed point, Implicit relation, Cauchy sequence, Intuitionistic fuzzy metric space.
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## 1 Introduction

The concept of fuzzy sets was introduced initially by Zadeh [31] in 1965. George and Veeramani [10] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [13]. In 1986 Atanassov [6] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1]. Recently, many authors have proved fixed point theorems involving intuitionistic fuzzy sets (see [7, 11, 14, 17–19, 28, 30] and references therein). Motivated by some work of V. Popa et al. via implicit relation, we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition. In this paper, we prove a related fixed point theorem for four mappings using an implicit relation. Our theorem generalizes the theorem of Popa [23] and other theorems in literature.

## 2 Preliminaries

For the terminologies and basic properties of intuitionistic fuzzy metric, we begin with some definitions, as follows.

**Definition 1** ( [26]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* $t$*-norm if it satisfies the following conditions:*

*1)* $*$ *is associative and commutative,*

*2)* $*$ *is continuous,*

*3)* $a * 1 = a$ *for all* $a \in [0,1]$*,*

*4)* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*; for each* $a, b, c, d \in [0,1]$.

*Two typical examples of a continuous t-norm are* $a * b = ab$ *and* $a * b = \min\{a, b\}$.

**Definition 2** ( [26]). *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-conorm if it satisfies the following conditions:*

*1)* $\Diamond$ *is associative and commutative,*

*2)* $\Diamond$ *is continuous,*

*3)* $a \Diamond 0 = a$ *for all* $a \in [0,1]$*,*

*4)* $a \Diamond b \leq c \Diamond d$ *whenever* $a \leq c$ *and* $b \leq d$*; for each* $a, b, c, d \in [0,1]$.

**Definition 3** ( [10]). *A* 3*-tuple* $(X, M, *)$ *is called a fuzzy metric space if* $X$ *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm and* $M$ *is a fuzzy metric on* $X^2 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*,*

*1)* $M(x, y, t) > 0$*,*

*2)* $M(x, y, t) = 1$ *if and only if* $x = y$*,*

*3)* $M(x, y, t) = M(y, x, t)$*,*

*4)* $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*,*

*5)* $M(x, y, .) : (0, \infty) \to [0,1]$ *is continuous.*

**Definition 4** ( [1]). *A* 5*-tuple* $(X, M, N, *, \Diamond)$ *is called an intuitionistic fuzzy metric space if* $X$ *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm,*$\Diamond$ *is a continuous t-conorm and* $M, N$ *are a fuzzy metric on* $X^2 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*,*

*1)* $M(x, y, t) + N(x, y, t) \leq 1$.

*2)* $M(x, y, t) > 0$*,*

*3)* $M(x, y, t) = 1$ *if and only if* $x = y$*,*

*4)* $M(x, y, t) = M(y, x, t)$*,*

*5)* $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*,*

*6)* $M(x, y, .) : (0, \infty) \to [0,1]$ *is continuous*

*7)* $M(x, y, t) > 0$*,*

*8)* $N(x, y, t) = 0$ *if and only if* $x = y$*,*

*9)* $N(x, y, t) = N(y, x, t)$*,*

*10)* $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$*,*

*11)* $N(x, y, .) : (0, \infty) \to [0,1]$ *is continuous.*

*Then $(M,N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x,y,t)$ and $N(x,y,t)$ respectively denote the degree of nearness and degree of nonnearness between $x$ and $y$ with respect to $t$.*

**Remark 1.** *In intuitionistic fuzzy metric space, $M(x,y,t)$ is non-decreasing and $N(x,y,t)$ is non-increasing for all $x,y \in X$.*

**Example 1** ( [27]). *Let $(X,d)$ be a metric space. Define t-norm $a*b=\min\{a,b\}$ and t-conorm $a\Diamond b=\max\{a,b\}$ and for all $x,y\in X$ and $t>0$,*

$$M(x,y,t)=\frac{t}{t+d(x,y)},\quad N(x,y,t)=\frac{d(x,y)}{t+d(x,y)}$$

*Then $(X,M,N,*,\Diamond)$ is an intuitionistic fuzzy metric space induced by the metric $d$. It is obvious that $N(x,y,t)=1-M(x,y,t)$.*

**Definition 5** ( [1]). *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space(IFM space), then*

*1) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x\in X$ if $\lim_{n\to\infty} M(x_n,x,t)=1$, $\lim_{n\to\infty} N(x_n,x,t)=0$; $\forall\, t>0$.*

*2) A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$, $\lim_{n\to\infty} N(x_{n+p},x_n,t)=0$;$\forall\, t>0$ and $p>0$.*

**Definition 6** ( [2]). *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space in which every Cauchy sequence is convergent, then $(X,M,N,*,\Diamond)$ is said to be a complete fuzzy metric space.*

**Lemma 1** ( [2]). *Let $\{u_n\}$ be a sequence in an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$. If there exists a constant $k\in(0,1)$ such that $M(u_n,u_{n+1},kt)\geq M(u_{n-1},u_n,t)$ and $N(u_n,u_{n+1},kt)\leq N(u_{n-1},u_n,t)$ for all $t>0$, then $\{u_n\}$ is a Cauchy sequence in $X$.*

**Lemma 2** ( [2]). *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space and for all $x,y\in X, t>0$ and if for a number $k\in(0,1)$, $M(x,y,kt)\geq M(x,y,t)$ and $N(x,y,kt)\leq N(x,y,t)$ then $x=y$.*

The following theorem is proved by V. Popa [23].

**Theorem 1.** *Let $(X,d)$ and $(Y,\rho)$ be complete metric spaces. Let $A,B$ be mappings of $X$ into $Y$ and let $S,T$ be mappings of $Y$ into $X$ satisfying the inequalities:*

$$F(d(SAx,TBx'),d(x,x'),d(x,SAx),d(x',TBx'),\rho(Ax,Bx'))\leq 0,$$

$$G(\rho(BSy,ATy'),\rho(y,y'),\rho(y,BSy),\rho(y',ATy'),d(Sy,Ty'))\leq 0,$$

*for all $x,x'$ in $X$ and $y,y'$ in $Y$, where $F,G\in\mathscr{F}_5$. If one of the mappings $A,B,T$ and $S$ is continuous, then $SA$ and $TB$ have a common fixed point $z$ in $X$ and $BS$ and $AT$ have a common fixed point $w$ in $Y$. Further, $Az=Bz=w$ and $Sw=Tw=z$.*

## 3 Implicit relation

Implicit relations play an important role in establishing of the fixed point theorems (see [3–5, 15, 20, 22]). Our implicit relation can be described as follows: Let $\Phi$ be the set of the functions $\varphi_1, \varphi_2, \theta_1, \theta_2 : [0,1]^5 \to \mathbb{R}$ such that, for every $u, v, w \in (0,1)$

$(H_1)$ $\varphi_1, \varphi_2, \theta_1, \theta_2$ are continuous in each coordinate variable,

$(H_2)$ $\varphi_1, \theta_1$ are nonincreasing in second and 3rd variables,

$(H_3)$ $\varphi_2, \theta_2$ is nonincreasing in second and 4th variables,

$(H_4)$ $\varphi_1 (u, v, u, v, w) \geq 0 \Rightarrow u \geq \min \{v, w\}$.

$(H_5)$ $\theta_1 (u, v, u, v, w) \leq 0 \Rightarrow u \leq \max \{v, w\}$.

$(H_6)$ $\varphi_2 (u, v, v, u, w) \geq 0 \Rightarrow u \geq \min \{v, w\}$.

$(H_7)$ $\theta_2 (u, v, v, u, w) \leq 0 \Rightarrow u \leq \max \{v, w\}$.

$(H_8)$ $\varphi_1 (u, 1, v, 1, 1) \geq 0$ or $\varphi_1 (u, u, 1, 1, v) \geq 0 \Rightarrow u \geq v$.

$(H_9)$ $\theta_1 (u, 0, v, 0, 0) \leq 0$ or $\theta_1 (u, u, 0, 0, v) \leq 0 \Rightarrow u \leq v$.

$(H_{10})$ $\varphi_2 (u, 1, v, 1, 1) \geq 0$ or $\varphi_2 (u, v, v, v, 1) \geq 0 \Rightarrow u \geq v$.

$(H_{11})$ $\theta_2 (u, 0, v, 0, 0) \leq 0$ or $\theta_2 (u, v, v, v, 0) \leq 0 \Rightarrow u \leq v$.

The above definitions, results and implicit relation motivated us to prove new related fixed point theorems for four mappings on intuitionistic fuzzy metric spaces by using implicit relation.

## 4 Main result

The main result of this paper is the following theorem.

**Theorem 2.** *Let $(X, M_1, N_1, *, \Diamond)$ and $(Y, M_2, N_2, *, \Diamond)$ be a complete intuitionistic fuzzy metric spaces with $M_1(x, x', t) \to 1$ as $t \to \infty$ for all $x, x' \in X$ and $M_2(y, y', t) \to 1$ as $t \to \infty$ for all $y, y' \in Y$. Let $A, B$ be mappings of $X$ into $Y$ and let $S, T$ be mappings of $Y$ into $X$ satisfying:*

$$\varphi_1(M_1(SAx, TBx', kt), M_1(x, x', t), M_1(x, SAx, t), M_1(x', TBx', t), M_2(Ax, Bx', t)) \geq 0 \tag{4.1}$$

$$\theta_1(N_1(SAx, TBx', kt), N_1(x, x', t), N_1(x, SAx, t), N_1(x', TBx', t), N_2(Ax, Bx', t)) \leq 0 \tag{4.2}$$

$$\varphi_2(M_2(BSy, ATy', kt), M_2(y, y', t), M_2(y, BSy, t), M_2(y', ATy', t), M_1(Sy, Ty', t)) \geq 0 \tag{4.3}$$

$$\theta_2(N_2(BSy, ATy', kt), N_2(y, y', t), N_2(y, BSy, t), N_2(y', ATy', t), N_1(Sy, Ty', t)) \leq 0 \tag{4.4}$$

*for all $x, x'$in $X$ and $y, y'$ in $Y$ and for all $t > 0$, where $\varphi_1, \varphi_2, \theta_1, \theta_2 \in \Phi$ and $0 < k < 1$. Then, if one of the mappings $A, B, T$ and $S$ is continuous then $SA$ and $TB$ have a unique fixed point $z$ in $X$ and $BS$ and $AT$ have a unique fixed point $w$ in $Y$. Further, $Az = Bz = w$ and $Sw = Tw = z$.*

*Proof.* Let $x$ be an arbitrary point in $X$. We define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$ respectively by:

$$Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}.$$

Using the inequality (4.1) and (4.2), we have successively

$$\varphi_1(M_1(SAx_{2n}, TBx_{2n-1}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, SAx_{2n}, t), M_1(x_{2n-1}, TBx_{2n-1}, t), M_2(Ax_{2n}, Bx_{2n-1}, t)) \geq 0,$$

$$\theta_1(N_1(SAx_{2n}, TBx_{2n-1}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, SAx_{2n}, t), N_1(x_{2n-1}, TBx_{2n-1}, t), N_2(Ax_{2n}, Bx_{2n-1}, t)) \leq 0,$$

that is,

$$\varphi_1(M_1(x_{2n+1}, x_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n-1}, x_{2n}, t), M_2(y_{2n}, y_{2n+1}, t)) \geq 0.$$

$$\theta_1(N_1(x_{2n+1}, x_{2n}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, x_{2n+1}, t), N_1(x_{2n-1}, x_{2n}, t), N_2(y_{2n}, y_{2n+1}, t)) \leq 0.$$

As $\varphi_1$ and $\theta_1$ are nonincreasing in the third variable, we get

$$\varphi_1(M_1(x_{2n+1}, x_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, x_{2n+1}, kt), M_1(x_{2n-1}, x_{2n}, t), M_2(y_{2n}, y_{2n+1}, t)) \geq 0.$$

$$\theta_1(N_1(x_{2n+1}, x_{2n}, kt), N_1(x_{2n}, x_{2n-1}, t), N_1(x_{2n}, x_{2n+1}, kt), N_1(x_{2n-1}, x_{2n}, t), N_2(y_{2n}, y_{2n+1}, t)) \leq 0.$$

which implies by $(H_4)$ and $(H_5)$ respectively

$$\begin{aligned} M_1(x_{2n+1}, x_{2n}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n}, y_{2n+1}, t)\}. && (4.5)\\ N_1(x_{2n+1}, x_{2n}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n}, y_{2n+1}, t)\}. && (4.6) \end{aligned}$$

Using inequality (4.1) and (4.2) again, it follows that

$$\begin{aligned} M_1(x_{2n}, x_{2n-1}, t) &\geq \min\{M_1(x_{2n-1}, x_{2n-2}, t), M_2(y_{2n}, y_{2n-1}, t)\}, && (4.7)\\ N_1(x_{2n}, x_{2n-1}, t) &\leq \max\{N_1(x_{2n-1}, x_{2n-2}, t), N_2(y_{2n}, y_{2n-1}, t)\} && (4.8) \end{aligned}$$

Similarly, using inequality (4.3) and (4.4), we get

$$\varphi_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, t), M_1(x_{2n}, x_{2n-1}, t)\right) \geq 0.$$

$$\theta_2\left(N_2(y_{2n+1}, y_{2n}, kt), N_2(y_{2n}, y_{2n-1}, t), N_2(y_{2n}, y_{2n-1}, t), N_2(y_{2n+1}, y_{2n}, t), N_1(x_{2n}, x_{2n-1}, t)\right) \leq 0.$$

Since $\varphi_2$ and $\theta_2$ are nonincreasing in the fourth variable, we obtain

$$\varphi_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t)\right) \geq 0.$$

$$\theta_2\left(M_2(y_{2n+1}, y_{2n}, kt), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, kt), M_1(x_{2n}, x_{2n-1}, t)\right) \leq 0.$$

From $(H_6)$ and $(H_7)$ respectevely, we have

$$\begin{aligned} M_2(y_{2n}, y_{2n+1}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n}, t)\}, && (4.9)\\ N_2(y_{2n}, y_{2n+1}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n-1}, y_{2n}, t)\} && (4.10) \end{aligned}$$

and

$$\begin{aligned} M_2(y_{2n-1}, y_{2n}, kt) &\geq \min\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-2}, y_{2n-1}, t)\}. && (4.11)\\ N_2(y_{2n-1}, y_{2n}, kt) &\leq \max\{N_1(x_{2n-2}, x_{2n-1}, t), N_2(y_{2n-2}, y_{2n-1}, t)\} && (4.12) \end{aligned}$$

Using inequalities (4.5), (4.9), and (4.6), (4.10) we have

$$\begin{aligned} M_1(x_{2n+1}, x_{2n}, kt) &\geq \min\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n}, t)\}, && (4.13)\\ N_1(x_{2n+1}, x_{2n}, kt) &\leq \max\{N_1(x_{2n}, x_{2n-1}, t), N_2(y_{2n-1}, y_{2n}, t)\}. && (4.14) \end{aligned}$$

Similarly, from inequalities (4.7), (4.11) and (4.8), (4.12), we have

$$M_1(x_{2n}, x_{2n-1}, t) \geq \min\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-2}, y_{2n-1}, t)\} \tag{4.15}$$
$$N_1(x_{2n}, x_{2n-1}, t) \leq \max\{N_1(x_{2n-2}, x_{2n-1}, t), N_2(y_{2n-2}, y_{2n-1}, t)\}. \tag{4.16}$$

It now follows from inequalities (4.13), (4.14) and (4.15), (4.16) that

$$M_1(x_{n+1}, x_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\} \tag{4.17}$$
$$N_1(x_{n+1}, x_n, kt) \leq \max\{N_1(x_n, x_{n-1}, t), N_2(y_n, y_{n-1}, t)\} \tag{4.18}$$

and

$$M_2(y_{n+1}, y_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\} \tag{4.19}$$
$$N_2(y_{n+1}, y_n, kt) \leq \max\{N_1(x_n, x_{n-1}, t), N_2(y_n, y_{n-1}, t)\}. \tag{4.20}$$

It now follows inequalities (4.17), (4.18) and (4.19), (4.20)

$$M_1(x_{n+1}, x_n, kt) \geq M_2(y_n, y_{n-1}, t) \tag{4.21}$$
$$M_2(y_{n+1}, y_n, kt) \geq M_1(x_n, x_{n-1}, t) \tag{4.22}$$

and

$$N_1(x_{n+1}, x_n, kt) \leq N_2(y_n, y_{n-1}, t) \tag{4.23}$$
$$N_2(y_{n+1}, y_n, kt) \leq N_1(x_n, x_{n-1}, t). \tag{4.24}$$

Using (4.21), (4.22) and (4.23), (4.24) we have for $n = 1, 2, \ldots$

$$M_1(x_{n+1}, x_n, t) \geq M_2\left(y_n, y_{n-1}, \frac{t}{k^2}\right)$$
$$M_2(y_{n+1}, x_n, t) \geq M_1\left(x_n, x_{n-1}, \frac{t}{k^2}\right)$$

and

$$N_1(x_{n+1}, x_n, t) \leq N_2\left(y_n, y_{n-1}, \frac{t}{k^2}\right)$$
$$N_2(y_{n+1}, y_n, t) \leq N_1\left(x_n, x_{n-1}, \frac{t}{k^2}\right).$$

For $n = 1, 2\ldots$, since $0 \leq k < 1$, from Lemma 2 it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$ and $Y$, respectively. Hence, $\{x_n\}$ converges to $z$ in $X$ and $\{y_n\}$ converges to $w$ in $Y$.

Now suppose that $A$ is continuous. Then

$$\lim Ax_{2n} = Az = \lim y_{2n+1} = w$$

and so $Az = w$. Using inequality (4.1) and (4.2) we get

$$\varphi_1\left(M_1(SAz, TBx_{2n-1}, kt), M_1(z, x_{2n-1}, t), M_1(z, SAz, t), M_1(x_{2n-1}, TBx_{2n-1}, t), M_2(Az, Bx_{2n-1}, t)\right) \geq 0,$$

$$\theta_1\left(N_1(SAz, TBx_{2n-1}, kt), N_1(z, x_{2n-1}, t), N_1(z, SAz, t), N_1(x_{2n-1}, TBx_{2n-1}, t), N_2(Az, Bx_{2n-1}, t)\right) \leq 0,$$

that is,

$$\varphi_1\left(M_1(Sw, x_{2n}, kt), M_1(z, x_{2n-1}, t), M_1(z, Sw, t), M_1(x_{2n-1}, x_{2n}, t), M_2(w, y_{2n}, t)\right) \geq 0.$$

$$\theta_1\left(N_1(Sw, x_{2n}, kt), N_1(z, x_{2n-1}, t), N_1(z, Sw, t), N_1(x_{2n-1}, x_{2n}, t), N_2(w, y_{2n}, t)\right) \leq 0.$$

Letting $n$ tend to infinity, we have

$$\begin{aligned} \varphi_1(M_1(Sw, z, kt), 1, M_1(z, Sw, t), 1, 1) &\geq 0, \\ \theta_1(N_1(Sw, z, kt), 0, N_1(z, Sw, t), 0, 0) &\leq 0 \end{aligned}$$

from $(H_8)$ and $(H_9)$, we get

$$\begin{aligned} M_1(Sw, z, kt) &\geq M_1(z, Sw, t), \\ N_1(Sw, z, kt) &\leq N_1(z, Sw, t), \end{aligned}$$

and so $Sw = z = SAz$. On the other hand, using inequality (4.3) and (4.4) we have successively

$$\varphi_2\left(M_2(BSw, ATy_{2n}, kt), M_2(w, y_{2n}, t), M_2(w, BSw, t), M_2(y_{2n}, ATy_{2n}, t), M_1(Sw, Ty_{2n}, t)\right) \geq 0,$$

$$\theta_2\left(N_2(BSw, ATy_{2n}, kt), N_2(w, y_{2n}, t), N_2(w, BSw, t), N_2(y_{2n}, ATy_{2n}, t), N_1(Sw, Ty_{2n}, t)\right) \geq 0,$$

then

$$\begin{aligned} \varphi_2\left(M_2(Bz, y_{2n+1}, kt), M_2(w, y_{2n}, t), M_2(w, Bz, t), M_2(y_{2n}, y_{2n+1}, t), M_1(z, x_{2n}, t)\right) &\geq 0. \\ \theta_2\left(N_2(Bz, y_{2n+1}, kt), N_2(w, y_{2n}, t), N_2(w, Bz, t), N_2(y_{2n}, y_{2n+1}, t), N_1(z, x_{2n}, t)\right) &\leq 0. \end{aligned}$$

Letting $n$ tend to infinity, we have

$$\begin{aligned} \varphi_2(M_2(Bz, w, kt), 1, M_2(w, Bz, t), 1, 1) &\geq 0 \\ \theta_2(N_2(Bz, w, kt), 0, N_2(w, Bz, t), 0, 0) &\leq 0 \end{aligned}$$

Thus, from $(H_{10})$ and $(H_{11})$, we get

$$\begin{aligned} M_2(Bz, w, kt) &\geq M_2(w, Bz, t) \\ N_2(Bz, w, kt) &\leq N_2(w, Bz, t), \end{aligned}$$

and so $w = Bz = BSw$. Using inequalities (4.1), (4.2) and (4.3), (4.4) we have respectively:

$$\begin{aligned} z &= Tw \text{ and } z = Tw = TBz \\ w &= ATw. \end{aligned}$$

The same result holds also if one of the mappings $B, S, T$ is continuous . To prove the uniqueness, suppose that $TB$ and $SA$ have a second distinct common fixed point $z'$. Then, using inequalities (4.1) and (4.2), we get

$$\begin{aligned} \varphi_1\left(M_1(SAz, TBz', kt), M_1(z, z', t), M_1(z, SAz, t), M_1(z', TBz', t), M_2(Az, Bz', t)\right)) &\geq 0 \\ \theta_1\left(N_1(SAz, TBz', kt), N_1(z, z', t), N_1(z, SAz, t), N_1(z', TBz', t), N_2(Az, Bz', t)\right) &\leq 0 \end{aligned}$$

that is

$$\begin{aligned} \varphi_1\left(M_1(z, z', kt), M_1(z, z', t), M_1(z, z, t), M_1(z', z', t), M_2(Az, Bz', t)\right)) &\geq 0 \\ \theta_1\left(N_1(z, z', kt), N_1(z, z', t), N_1(z, z, t), N_1(z', z', t), N_2(Az, Bz', t)\right) &\leq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi_1\left(M_1(z, z', kt), M_1(z, z', t), 1, 1, M_2(Az, Bz', t)\right)) &\geq 0 \\ \theta_1\left(N_1(z, z', kt), N_1(z, z', t), 0, 0, N_2(Az, Bz', t)\right) &\leq 0 \end{aligned}$$

As $\varphi_1$ and $\theta_1$ are nonincreasing in second variable, we get

$$\begin{aligned} \varphi_1\left(M_1(z, z', kt), M_1(z, z', kt), 1, 1, M_2(Az, Bz', t)\right)) &\geq 0 \\ \theta_1\left(N_1(z, z', kt), N_1(z, z', kt), 0, 0, N_2(Az, Bz', t)\right) &\leq 0, \end{aligned}$$

which implies by $(H_8)$ and $(H_9)$

$$\begin{aligned} M_1(z, z', kt) &\geq M_2(Az, Bz', t) && (4.25) \\ N_1(z, z', kt) &\leq N_2(Az, Bz', t). && (4.26) \end{aligned}$$

Further, applying inequalities (4.3) and (4.4), we obtain

$$\begin{aligned} \varphi_2\left(M_2(Bz', Az, kt), M_2(Az, Bz', t), M_2(Az, Bz', t), M_2(Bz', Az, t), M_1(z', z, t)\right) &\geq 0 \\ \theta_2\left(N_2(Bz', Az, kt), N_2(Az, y', t), N_2(Az, Bz', t), N_2(Bz', Az, t), N_1(z', z, t)\right) &\leq 0. \end{aligned}$$

Using $(H_{10})$ and $(H_{11})$, we get

$$\begin{aligned} M_2(Bz', Az, kt) &\geq M_1(z', z, t) && (4.27) \\ N_2(Bz', Az, kt) &\leq N_1(z', z, t). && (4.28) \end{aligned}$$

By (4.25), (4.26) and (4.27), (4.28)

$$\begin{aligned} M_1(z, z', kt) &\geq M_1(z', z, t) \\ N_1(z, z', kt) &\leq N_1(z', z, t). \end{aligned}$$

Therefore, contradiction with Lemma 2, then, the pair $TB$ and $SA$ have a unique common fixed point. The uniqueness of $w$ follows in a similar manner. The proof is complete. □

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