Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 25, 2019, No. 3, 13–25
DOI: 10.7546/nifs.2019.25.3.13-25

Four interval-valued intuitionistic fuzzy modal-level operators

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia 1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Burgas 8010, Bulgaria
e-mail: krat@bas.bg

In memory of my friend and colleague
Prof. Magdalina Todorova (1954–2019)

Received: 6 June 2019 **Accepted:** 10 August 2019

Abstract: Four new interval-valued intuitionistic fuzzy operators are introduced. It is shown for them that they exhibit behaviour similar both to the modal, as well as to the level operators defined over interval-valued intuitionistic fuzzy sets, and for this reason, they are called interval-valued intuitionistic fuzzy modal-level operators. Their basic properties are discussed.
Keywords: Interval-valued intuitionistic fuzzy set, Interval-valued intuitionistic fuzzy operator.
2010 Mathematics Subject Classification: 03E72.

1 Introduction

The Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) were introduced exactly 30 years ago in [10] and over the years their theory has been enriched with a lot of operators that do not have analogues in the theories of standard fuzzy sets, intuitionistic fuzzy sets (IFSs), as well as the rest of the fuzzy sets extensions. In the present paper, we introduce three operators that exhibit behaviour similar to the modal, as well as to the level operators.

2 Preliminary definitions

Following [4, 5], we give the definitions of the basic concepts and the basic operations, relations and operators over IFSs.

Let us have a fixed universe E and its subset A. The set

$$A^* = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where $M_A(x) \subseteq [0,1]$ and $N_A(x) \subseteq [0,1]$ are intervals and for all $x \in E$ such that:

$$\sup M_A(x) + \sup N_A(x) \leq 1,$$

is called an Interval-Valued Intuitionistic Fuzzy Set (IVIFS) and the intervals $M_A(x), N_A(x)$ represent the *interval of the degree of membership (validity, etc.)* and the *interval of the degree of non-membership (non-validity, etc.),* respectively.

Obviously, this definition is constructed analogously to the definition of an IFS (cf. [4, 5]).

IVIFSs have geometrical interpretations similar to–but more complex than–these of the IFSs (see Fig. 1).

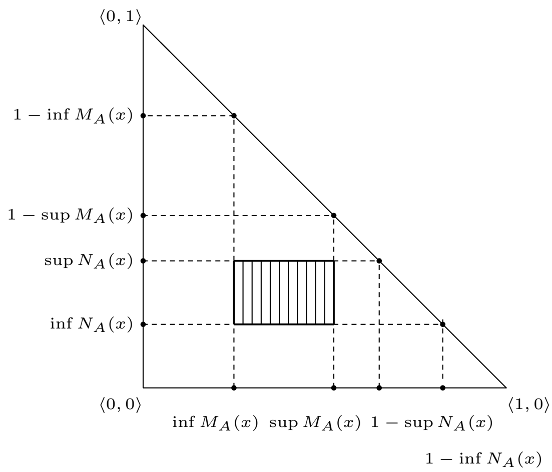

Figure 1. Geometrical interpretation of an IVIFS element x

For brevity, everywhere hereafter we write A instead of A^*.

Following [4], we introduce some relations and operations for two IVIFSs A and B:

$$\begin{array}{lcl}
A \subset B & \text{iff} & (\forall x \in E)(\inf M_A(x) \leq \inf M_B(x) \;\&\; \sup M_A(x) \leq \sup M_B(x)), \\
& & \quad \& \inf N_A(x) \geq \inf N_B(x) \;\&\; \sup N_A(x) \geq \sup N_B(x)), \\
A = B & \text{iff} & (\forall x \in E)(\inf M_A(x) = \inf M_B(x) \;\&\; \sup M_A(x) = \sup M_B(x), \\
& & \quad \& \inf N_A(x) = \inf N_B(x) \;\&\; \sup N_A(x) = \sup N_B(x)); \\
\neg A & = & \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}, \\
A \cap B & = & \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\
& & \quad [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\},
\end{array}$$

$$
\begin{aligned}
A \cup B \;&=\; \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))], \\
&\qquad [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}, \\
A + B \;&=\; \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x). \inf M_B(x), \\
&\qquad\quad \sup M_A(x) + \sup M_B(x) - \sup M_A(x). \sup M_B(x)], \\
&\qquad [\inf N_A(x). \inf N_B(x), \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\}, \\
A.B \;&=\; \{\langle x, [\inf M_A(x). \inf M_B(x), \sup M_A(x). \sup M_B(x)], \\
&\qquad [\inf N_A(x) + \inf N_B(x) - \inf N_A(x). \inf N_B(x), \\
&\qquad\quad \sup N_A(x) + \sup N_B(x) - \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\}, \\
A@B \;&=\; \left\{\left\langle x, \left[\tfrac{\inf M_A(x)+\inf M_B(x)}{2}, \tfrac{\sup M_A(x)+\sup M_B(x)}{2}\right],\right.\right. \\
&\qquad \left.\left.\left[\tfrac{\inf N_A(x)+\inf N_B(x)}{2}, \tfrac{\sup N_A(x))+\sup N_B(x)}{2}\right]\right| x \in E\right\}.
\end{aligned}
$$

3 The new operators

Now, we introduce the first two new operators, defined over a given IVIFS A. They have the forms:

$$
\begin{aligned}
\widetilde{H}_{\alpha,\beta}(A) =&\{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], \\
&\qquad [\inf N_A(x) + \beta - \beta \inf N_A(x), \sup N_A(x) + \beta - \beta \sup N_A(x)]\rangle | x \in E\} \\
\widetilde{J}_{\alpha,\beta}(A) =&\{\langle x, [\inf M_A(x) + \alpha - \alpha \inf M_A(x), \sup M_A(x) + \alpha - \alpha \sup M_A(x)], \\
&\qquad [\beta \inf N_A(x), \beta \sup N_A(x)]\rangle | x \in E\},
\end{aligned}
$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

We check that

$$0 \leq \alpha \inf M_A(x) \leq \alpha \sup M_A(x) \leq 1,$$

$$0 \leq \inf N_A(x) + \beta - \beta \inf N_A(x) \leq \sup N_A(x) + \beta - \beta \sup N_A(x) \leq 1,$$

$$Z \equiv \alpha \sup M_A(x) + \sup N_A(x) + \beta - \beta \sup N_A(x)$$

$$\leq \alpha(1 - \sup N_A(x)) + \sup N_A(x) + \beta(1 - \sup N_A(x))$$

$$\leq \alpha + \beta + (1 - \alpha - \beta) \sup N_A(x) \leq \alpha + \beta + 1 - \alpha - \beta = 1$$

i.e., the first definition is correct. Analogously, we check that the second definition is also correct.

After this, we see that for each IVIFS A and for every $\alpha, \beta, \gamma, \delta, \in [0, 1]$, such that $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$:

$$\widetilde{H}_{\alpha,\beta}(A) \subseteq A \subseteq \widetilde{J}_{\gamma,\delta}(A)$$

and the inequalities are transformed to equalities if and only if $\alpha = \delta = 1$ and hence $\beta = \gamma = 0$.

Similarly to the rest modal operators, defined over IVIFSs, the two new operators can be also extended to the forms

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)], [\inf N_A(x) + \gamma - \gamma \inf N_A(x), \sup N_A(x) + \delta - \delta \sup N_A(x)\rangle | x \in E\},$$

$$\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\inf M_A(x) + \alpha - \alpha \inf M_A(x), \sup M_A(x) + \beta - \beta \sup M_A(x)], [\gamma \inf N_A(x), \delta \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0,1]$, $\alpha \leq \beta, \gamma \leq \delta, \beta + \delta \leq 1$.

We can see again that both operators are correctly defined and for them the inequalities

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) \subseteq A \subseteq \widetilde{J}_{\left(\begin{smallmatrix}\alpha' & \gamma' \\ \beta' & \delta'\end{smallmatrix}\right)}(A)$$

hold for every $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in [0,1]$, such that $\alpha \leq \beta, \gamma \leq \delta, \beta + \delta \leq 1, \alpha' \leq \beta', \gamma' \leq \delta', \beta' + \delta' \leq 1$.

Obviously, for each IVIFS A:

$$\widetilde{H}_{\alpha,\beta}(A) = \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \beta \\ \alpha & \beta\end{smallmatrix}\right)}(A),$$

$$\widetilde{J}_{\alpha,\beta}(A) = \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \beta \\ \alpha & \beta\end{smallmatrix}\right)}(A).$$

By this reason, hereafter, we will only work with the two extended operators.

Theorem 1. For each IVIFS A and for $\alpha, \beta, \gamma, \delta \in [0,1]$, such that $\alpha \leq \beta, \gamma \leq \delta, \beta + \delta \leq 1$:

$$\neg \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(\neg A) = \widetilde{J}_{\left(\begin{smallmatrix}\gamma & \delta \\ \alpha & \beta\end{smallmatrix}\right)}(A),$$

$$\neg \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(\neg A) = \widetilde{H}_{\left(\begin{smallmatrix}\gamma & \delta \\ \alpha & \beta\end{smallmatrix}\right)}(A).$$

Theorem 2. For each IVIFS A and for $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in [0,1]$, such that $\alpha \leq \beta, \gamma \leq \delta$, $\beta + \delta \leq 1$ and $\alpha' \leq \beta', \gamma' \leq \delta', \beta' + \delta' \leq 1$:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(\widetilde{J}_{\left(\begin{smallmatrix}\alpha' & \gamma' \\ \beta' & \delta'\end{smallmatrix}\right)}(A)) \subseteq \widetilde{J}_{\left(\begin{smallmatrix}\alpha' & \gamma' \\ \beta' & \delta'\end{smallmatrix}\right)}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A)).$$

Two analogues of the topological operators have been defined over the IVIFSs (see, e.g., [4]): operator *"closure"* C and operator *"interior"* I:

$$C(A) = \{\langle x, [K'_{\inf}, K'_{\sup}], [L'_{\inf}, L'_{\sup}]\rangle \mid x \in E\},$$

$$I(A) = \{\langle x, [K''_{\inf}, K''_{\sup}], [L''_{\inf}, L''_{\sup}]\rangle \mid x \in E\},$$

where:

$$
\begin{aligned}
K'_{\inf} = &\ \sup_{y\in E}\inf M_A(y), \quad K'_{\sup} = &\ \sup_{y\in E}\sup M_A(y),\\
L'_{\inf} = &\ \inf_{y\in E}\inf N_A(y), \quad L'_{\sup} = &\ \inf_{y\in E}\sup N_A(y),\\
K''_{\inf} = &\ \inf_{y\in E}\inf M_A(y), \quad K''_{\sup} = &\ \inf_{y\in E}\sup M_A(y),\\
L''_{\inf} = &\ \sup_{y\in E}\inf N_A(y), \quad L''_{\sup} = &\ \sup_{y\in E}\sup N_A(y).
\end{aligned}
$$

Theorem 3. For every two IFSs A and B, and for every $\alpha,\beta,\gamma,\delta\in[0,1]$, such that $\alpha\le\beta$, $\gamma\le\delta,\beta+\delta\le 1$:

$$
\begin{aligned}
C(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)) &= \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(C(A)),\\
I(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)) &= \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(I(A)),\\
C(\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)) &= \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(C(A)),\\
I(\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)) &= \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(I(A)).
\end{aligned}
$$

Proof. Let A be a given IVIFS and let $\alpha,\beta,\gamma,\delta\in[0,1]$. Then

$$
\begin{aligned}
C(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)) &= C(\{\langle x,[\alpha\inf M_A(x),\gamma\sup M_A(x)],\\
&[\inf N_A(x)+\beta-\beta\inf N_A(x),\sup N_A(x)+\delta-\delta\sup N_A(x)]\rangle | x\in E\})\\
&= \{\langle x,[\sup_{y\in E}\alpha\inf M_A(y),\sup_{y\in E}\gamma\sup M_A(y)],\\
&[\inf_{y\in E}(\inf N_A(y)+\beta-\beta\inf N_A(y)),\inf_{y\in E}(\sup N_A(y)+\delta-\delta\sup N_A(y))\rangle | x\in E\}\\
&= \{\langle x,[\alpha\sup_{y\in E}\inf M_A(y),\gamma\sup_{y\in E}\sup M_A(y)],\\
&[\beta+\inf_{y\in E}((1-\beta)\inf N_A(y)),\delta+\inf_{y\in E}((1-\delta)\sup N_A(y))\rangle | x\in E\}\\
&= \{\langle x,[\alpha\sup_{y\in E}\inf M_A(y),\gamma\sup_{y\in E}\sup M_A(y)],\\
&[\beta+(1-\beta)\inf_{y\in E}\inf N_A(y),\delta+(1-\delta)\inf_{y\in E}\sup N_A(y)\rangle | x\in E\}\\
&= \{\langle x,[\alpha\sup_{y\in E}\inf M_A(y),\gamma\sup_{y\in E}\sup M_A(y)],\\
&[\inf_{y\in E}\inf N_A(y)+\beta-\beta\inf_{y\in E}\inf N_A(y),\inf_{y\in E}\sup N_A(y)+\delta-\delta\inf_{y\in E}\sup N_A(y)]\rangle | x\in E\}\\
&= \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\{\langle x,[\sup_{y\in E}\inf M_A(y),\sup_{y\in E}\sup M_A(y)],\\
&[\inf_{y\in E}\inf N_A(y),\inf_{y\in E}\sup N_A(y)\rangle | x\in E\})\\
&= \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(C(A)).
\end{aligned}
$$

The other assertions are proved in the same way. □

Below, we will discuss the four interval-valued intuitionistic fuzzy operators from three different perspectives.

3.1 The perspective of the interval-valued intuitionistic fuzzy modal operators of first type

The simplest intuitionistic fuzzy modal operators are analogous of modal operators "necessity" and "possibility". In the framework of the IVIFSs theory these operators are extended and modified in a "step-by-step" manner. The first group of modal operators is the following (see [4, 7]):

$$\begin{aligned}
\square A = {} & \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\}, \\
\Diamond A = {} & \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\}, \\
D_\alpha(A) = {} & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\
& \quad [\inf N_A(x), \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \\
& \quad \mid x \in E\}, \\
F_{\alpha,\beta}(A) = {} & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\
& \quad [\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \\
& \quad \mid x \in E\}, \text{ for } \alpha + \beta \leq 1, \\
G_{\alpha,\beta}(A) = {} & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\}, \\
H_{\alpha,\beta}(A) = {} & \{\langle x, [\alpha.\inf M(x), \alpha.\sup M_A(x)], \\
& \quad [\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}, \\
H^*_{\alpha,\beta}(A) = {} & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], \\
& \quad [\inf N_A(x), \sup N_A(x) + \beta.(1 - \alpha.\sup M_A(x) - \sup N_A(x))]\rangle \\
& \quad \mid x \in E\}, \\
J_{\alpha,\beta}(A) = {} & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], \\
& \quad [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\}, \\
J^*_{\alpha,\beta}(A) = {} & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \beta.\sup N_A(x))], \\
& \quad [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\}, \\
\overline{F}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = {} & \{\langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \qquad \sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))], \\
& \quad [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \qquad \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E, \} \\
& \qquad \text{for } \beta + \delta \leq 1, \\
\overline{G}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = {} & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)], [\gamma.\inf N_A(x), \delta.\sup N_A(x)]\rangle \mid x \in E\}, \\
\overline{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = {} & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)], \\
& \quad [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \qquad \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},
\end{aligned}$$

$$\overline{H}^*_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)],$$
$$[\inf N_A(x) + \gamma.(1 - \beta.\sup M_A(x) - \sup N_A(x)),$$
$$\sup N_A(x) + \delta.(1 - \beta.\sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$
$$\overline{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)),$$
$$\sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))],$$
$$[\gamma.\inf N_A(x), \delta.\sup N_A(x)]\rangle \mid x \in E\},$$
$$\overline{J}^*_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\inf M_A(x) + \alpha.(1 - \delta.\sup M_A(x) - \sup N_A(x)),$$
$$\sup M_A(x) + \beta.(1 - \sup M_A(x) - \delta.\sup N_A(x))],$$
$$[\gamma.\inf N_A(x), \delta.\sup N_A(x)]\rangle \mid x \in E\},$$

for every $\alpha, \beta, \gamma, \delta \in [0, 1]$.

The composition of two $\square$-, $\diamondsuit$-, D-, F- and G-operators can be represented by only one of them, while this is impossible for the rest of the operators, but now, we see that the following assertion is valid.

Theorem 4. For each IVIFS A and for $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in [0, 1]$, such that $\alpha \leq \beta$, $\gamma \leq \delta, \beta + \delta \leq 1$ and $\alpha' \leq \beta', \gamma' \leq \delta', \beta' + \delta' \leq 1$:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha' & \gamma'\\ \beta' & \delta'\end{smallmatrix}\right)}(A)) = \widetilde{H}_{\left(\begin{smallmatrix}\alpha\alpha' & \gamma + \gamma' - \gamma\gamma'\\ \beta\beta' & \delta + \delta' - \delta\delta'\end{smallmatrix}\right)}(A),$$

$$\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\widetilde{J}_{\left(\begin{smallmatrix}\alpha' & \gamma'\\ \beta' & \delta'\end{smallmatrix}\right)}(A)) = \widetilde{J}_{\left(\begin{smallmatrix}\alpha + \alpha' - \alpha\alpha' & \gamma\gamma'\\ \beta + \beta' - \beta\beta' & \delta\delta'\end{smallmatrix}\right)}(A).$$

Proof: We check sequentially:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha' & \gamma'\\ \beta' & \delta'\end{smallmatrix}\right)}(A)) = \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\{\langle x, [\alpha' \inf M_A(x), \beta' \sup M_A(x)],$$
$$[\inf N_A(x) + \gamma' - \gamma' \inf N_A(x), \sup N_A(x) + \delta' - \delta' \sup N_A(x)\rangle | x \in E\})$$
$$= \{\langle x, [\alpha\alpha' \inf M_A(x), \beta\beta' \sup M_A(x)],$$
$$[\inf N_A(x) + \gamma' - \gamma' \inf N_A(x) + \gamma - \gamma(\inf N_A(x) + \gamma' - \gamma' \inf N_A(x)),$$
$$\sup N_A(x) + \delta' - \delta' \sup N_A(x) + \delta - \delta(\sup N_A(x) + \delta' - \delta' \sup N_A(x))\rangle | x \in E\})$$
$$= \{\langle x, [\alpha\alpha' \inf M_A(x), \beta\beta' \sup M_A(x)],$$
$$[\inf N_A(x) + \gamma + \gamma' - \gamma\gamma' - (\gamma + \gamma' - \gamma\gamma') \inf N_A(x),$$
$$\sup N_A(x) + \delta + \delta' - \delta\delta' - (\delta + \delta' - \delta\delta') \sup N_A(x)\rangle | x \in E\})$$
$$= \widetilde{H}_{\left(\begin{smallmatrix}\alpha\alpha' & \gamma + \gamma' - \gamma\gamma'\\ \beta\beta' & \delta + \delta' - \delta\delta'\end{smallmatrix}\right)}(A).$$

The remaining assertion as well as the below ones can be proved similarly. $\square$

Therefore, we see that these two operators satisfy the property of F- and G-operators.

Moreover, for each IVIFS A:

$$\widetilde{H}_{0,1}(A) = \widetilde{H}_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)}(A) = \{\langle x, [0,0], [1,1]\rangle | x \in E\} = O^*,$$

$$\widetilde{J}_{1,0}(A) = \widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(A) = \{\langle x, [1,1], [0,0]\rangle | x \in E\} = E^*,$$

similarly to operators $H^*_{0,1}$ and $J^*_{1,0}$, respectively.

Theorem 5. For every two IFSs A and B, and for every $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$:

$$\widetilde{H}_{\alpha,\beta}(A \cap B) = \widetilde{H}_{\alpha,\beta}(A) \cap \widetilde{H}_{\alpha,\beta}(B),$$

$$\widetilde{H}_{\alpha,\beta}(A \cup B) = \widetilde{H}_{\alpha,\beta}(A) \cup \widetilde{H}_{\alpha,\beta}(B),$$

$$\widetilde{J}_{\alpha,\beta}(A \cap B) = \widetilde{J}_{\alpha,\beta}(A) \cap \widetilde{J}_{\alpha,\beta}(B),$$

$$\widetilde{J}_{\alpha,\beta}(A \cup B) = \widetilde{J}_{\alpha,\beta}(A) \cup \widetilde{J}_{\alpha,\beta}(B),$$

$$\widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A \cap B) = \widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) \cap \widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(B),$$

$$\widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A \cup B) = \widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) \cup \widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(B),$$

$$\widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(A \cap B) = \widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(A) \cap \widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(B),$$

$$\widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(A \cup B) = \widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(A) \cup \widetilde{J}_{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)}(B).$$

Let $\text{ext}_1, \text{ext}_2, \text{ext}_3, \text{ext}_4, \text{ext}_5, \text{ext}_6, \text{ext}_7, \text{ext}_8 \in \{\inf, \sup\}$.

The most extended intuitionistic fuzzy modal operator from the first type has the form:

$$X^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \\ \text{ext}_5 & \text{ext}_6 & \text{ext}_7 & \text{ext}_8 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}(A)$$

$$\equiv \{\langle x, [\inf M_X(x), \sup M_X(x)], [\inf N_X(x), \sup N_X(x)]\rangle | x \in E\},$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \text{ext}_1 M_A(x) - c_1 \text{ext}_2 N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \text{ext}_3 M_A(x) - c_2 \text{ext}_4 N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \text{ext}_5 M_A(x) - \text{ext}_6 N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \text{ext}_7 M_A(x) - \text{ext}_8 N_A(x))]\rangle | x \in E\},$$

where $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0,1]$ and

$$0 \leq \inf M_X(x) \leq \sup M_X(x) \leq 1,$$

$$0 \leq \inf N_X(x) \leq \sup N_X(x) \leq 1,$$

$$\sup M_X(x) + \sup N_X(x) \leq 1.$$

Now, we see directly that

$$\widetilde{H}_{\alpha,\beta}(A) = X^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \\ \text{ext}_5 & \inf & \text{ext}_6 & \sup \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} \alpha & 0 & r_1 & 1 & \beta & 0 \\ \alpha & 0 & r_2 & 1 & \beta & 0 \end{smallmatrix}\right)}(A),$$

$$\widetilde{J}_{\alpha,\beta}(A) = X^{\left(\begin{smallmatrix} \inf & \text{ext}_1 & \sup & \text{ext}_2 \\ \text{ext}_3 & \text{ext}_4 & \text{ext}_5 & \text{ext}_6 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} 1 & \alpha & 0 & \beta & 0 & r_3 \\ 1 & \alpha & 0 & \beta & 0 & r_4 \end{smallmatrix}\right)}(A),$$

$$\widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) = X^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 & \text{ext}_3 & \text{ext}_4 \\ \text{ext}_5 & \inf & \text{ext}_6 & \sup \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} \alpha & 0 & r_1 & 1 & \gamma & 0 \\ \beta & 0 & r_2 & 1 & \delta & 0 \end{smallmatrix}\right)}(A),$$

$$\widetilde{J}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) = X^{\left(\begin{smallmatrix} \inf & \text{ext}_1 & \sup & \text{ext}_2 \\ \text{ext}_3 & \text{ext}_4 & \text{ext}_5 & \text{ext}_6 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} 1 & \alpha & 0 & \gamma & 0 & r_3 \\ 1 & \beta & 0 & \delta & 0 & r_4 \end{smallmatrix}\right)}(A),$$

where $r_1, r_2, r_3, r_4 \in [0, 1]$ are arbitrary numbers and $\text{ext}_1, \text{ext}_2, \text{ext}_3, \text{ext}_4, \text{ext}_5, \text{ext}_6 \in \{\inf, \sup\}$ are one of the two symbols, regardless of which.

Therefore, the new operators have a similar X-representation as the rest of the extended modal-type operators.

3.2 The perspective of the interval-valued intuitionistic fuzzy modal operators of second type

The Interval-Valued Intuitionistic Fuzzy Modal Operators of Second Type (IVIFMO2) are introduced for the first time in [8, 9]. They also have two forms: shorter (introduced in [8]) and extended (introduced in [9]). These IVIFO2s are represented by one – the most extended operator that has the form given below.

Let again $\text{ext}_1, \text{ext}_2 \in \{\inf, \sup\}$. We define

$$\boxed{\circ}^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \zeta_2 \end{smallmatrix}\right)} A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \text{ext}_1 N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2],$$
$$[\beta_1 \inf N_A(x) - \zeta_1 \text{ext}_1 M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2]\rangle | x \in E\}.$$

The components of this operator must satisfy the following conditions in a general form:

$$0 \leq \alpha_1 \inf M_A(x) - \varepsilon_1 \text{ext}_1 N_A(x) + \gamma_1 \leq \alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2 \leq 1,$$

$$0 \leq \beta_1 \inf N_A(x) - \zeta_1 \text{ext}_1 M_A(x) + \delta_1 \leq \beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2 \leq 1,$$

$$\alpha_2 \sup M_A(x) - \varepsilon_2 \text{ext}_2 N_A(x) + \gamma_2 + \beta_2 \sup N_A(x) - \zeta_2 \text{ext}_2 M_A(x) + \delta_2 \leq 1.$$

The two new (more extended) modal operators have the following representations:

$$\widetilde{H}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) = \boxed{\circ}^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} \alpha & 1-\gamma & 0 & \gamma & 0 & 0 \\ \beta & 1-\delta & 0 & \delta & 0 & 0 \end{smallmatrix}\right)} A,$$

$$\widetilde{J}_{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)}(A) = \boxed{\circ}^{\left(\begin{smallmatrix} \text{ext}_1 & \text{ext}_2 \end{smallmatrix}\right)}_{\left(\begin{smallmatrix} 1-\alpha & \gamma & \alpha & 0 & 0 & 0 \\ 1-\beta & \delta & \beta & 0 & 0 & 0 \end{smallmatrix}\right)} A,$$

where $\text{ext}_1, \text{ext}_2 \in \{\inf, \sup\}$.

3.3 The perspective of the interval-valued intuitionistic fuzzy level operators

The basic Interval-Valued Intuitionistic Fuzzy Level Operators (IVIFLO) are (see, e.g., [4]:

$$\begin{aligned}\overline{P}_{\alpha,\beta,\gamma,\delta} = \ & \{\langle x, [\max(\alpha, \inf M_A(x)), \max(\beta, \sup M_A(x))], \\ & \quad [\min(\gamma, \inf N_A(x)), \min(\delta, \sup N_A(x))]\rangle \mid x \in E\}, \\ \overline{Q}_{\alpha,\beta,\gamma,\delta} = \ & \{\langle x, [\min(\alpha, \inf M_A(x)), \min(\beta, \sup M_A(x))], \\ & \quad [\max(\gamma, \inf N_A(x)), \max(\delta, \sup N_A(x))]\rangle \mid x \in E\},\end{aligned}$$

for $\alpha, \beta, \gamma, \delta \in [0, 1]$,$\alpha \leq \beta$, $\gamma \leq \delta$ and $\beta + \delta \leq 1$.

The intervals of the degrees of membership and non-membership of the elements of a given universe to its subset can be directly changed by these operators.

Obviously, for every IVIFS A and for $\alpha, \beta, \gamma, \delta \in [0, 1]$, $\alpha \leq \beta$, $\gamma \leq \delta$ and $\beta + \delta \leq 1$:

$$\overline{P}_{\alpha,\beta,\gamma,\delta}(A) = A \cup \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\},$$

$$\overline{Q}_{\alpha,\beta,\gamma,\delta}(A) = A \cap \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\},$$

$$\overline{Q}_{\alpha,\beta,\gamma,\delta}(A) \subset A \subset \overline{P}_{\alpha,\beta,\gamma,\delta}(A).$$

Therefore, it will be suitable to denote both operators as follows:

$$O^{\cup}_{\alpha,\beta,\gamma,\delta}(A) = A \cup \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\},$$

$$O^{\cap}_{\alpha,\beta,\gamma,\delta}(A) = A \cap \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\}.$$

Now, we can define the two new intuitionistic fuzzy level operators on the basis of operations "$+$" and ".", defined in Section 2, but here, we use notation "$\times$" instead of ".":

$$O^{+}_{\alpha,\beta,\gamma,\delta}(A) = A + \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\},$$

$$O^{\times}_{\alpha,\beta,\gamma,\delta}(A) = A.\{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\}.$$

We see immediately that:

$$O^{+}_{\alpha,\beta,\gamma,\delta}(A) = \widetilde{J}_{\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}}(A),$$

$$O^{\times}_{\alpha,\beta,\gamma,\delta}(A) = \widetilde{H}_{\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}}(A),$$

while

$$\widetilde{H}_{\alpha,\beta}(A) = O^{\times}_{\alpha,\alpha,\beta,\beta}(A),$$

$$\widetilde{J}_{\gamma,\delta}(A) = O^{+}_{\alpha,\alpha,\beta,\beta}(A).$$

Therefore, the four new operators have the behaviour of level operators.

Theorem 6. For every IVIFS A, for $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta \in [0, 1]$, such that $\alpha \leq \beta, \gamma \leq \delta, \beta + \delta \leq 1, \varepsilon \leq \zeta, \eta \leq \theta, \zeta + \theta \leq 1$:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\overline{P}_{\varepsilon,\zeta,\eta,\theta}(A)) = \overline{P}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A))$$

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\overline{Q}_{\varepsilon,\zeta,\eta,\theta}(A)) = \overline{Q}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A))$$

$$\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\overline{P}_{\varepsilon,\zeta,\eta,\theta}(A)) = \overline{P}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A))$$

$$\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\overline{Q}_{\varepsilon,\zeta,\eta,\theta}(A)) = \overline{Q}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A))$$

Proof: Let the IVIFS A and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$ be given, so that they satisfy the above conditions. Then, for the first equality we obtain:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\overline{P}_{\varepsilon,\zeta,\eta,\theta}(A))$$

$$\begin{gathered}
= \widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(\{\langle x, [\max(\varepsilon, \inf M_A(x)), \max(\zeta, \sup M_A(x))],\\
[\min(\eta, \inf N_A(x)), \min(\theta, \sup N_A(x))]\rangle \mid x \in E\})\\
= \{\langle x, [\alpha \max(\varepsilon, \inf M_A(x)), \beta \max(\zeta, \sup M_A(x))],\\
[\min(\eta, \inf N_A(x)) + \gamma - \gamma \min(\eta, \inf N_A(x)),\\
\min(\theta, \sup N_A(x)) + \delta - \delta \min(\theta, \sup N_A(x))]\rangle \mid x \in E\}\\
= \{\langle x, [\max(\alpha\varepsilon, \alpha \inf M_A(x)), \max(\beta\zeta, \beta \sup M_A(x))],\\
[\min((1-\gamma)\eta + \gamma, (1-\gamma) \inf N_A(x) + \gamma),\\
\min((1-\delta)\theta + \delta, (1-\delta) \sup N_A(x) + \delta)]\rangle \mid x \in E\}\\
= \overline{P}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)],\\
[\inf N_A(x) + \gamma - \gamma \inf N_A(x), \sup N_A(x) + \delta - \delta \sup N_A(x)\rangle | x \in E\})\\
= \overline{P}_{\alpha\varepsilon,\beta\zeta,(1-\gamma)\eta+\gamma,(1-\delta)\theta+\delta}(\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)).
\end{gathered}$$

It can be directly seen that for each IVIFS A and for $\alpha, \beta, \gamma, \delta \in [0, 1]$, $\alpha \leq \beta$, $\gamma \leq \delta$ and $\beta + \delta \leq 1$:

$$\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) \subseteq \overline{Q}_{\alpha,\beta,\gamma,\delta}(A) \subseteq A \subseteq \overline{P}_{\alpha,\beta,\gamma,\delta}(A) \subseteq \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A).$$

Now, we can calculate for every IVIFS A and for $\alpha, \beta, \gamma, \delta \in [0, 1]$, $\alpha \leq \beta$, $\gamma \leq \delta$ and $\beta + \delta \leq 1$:

$$
\begin{aligned}
&\widetilde{H}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) @ \widetilde{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A)\\
&= \{\langle x, [\alpha \inf M_A(x), \beta \sup M_A(x)],\\
&[\inf N_A(x) + \gamma - \gamma \inf N_A(x), \sup N_A(x) + \delta - \delta \sup N_A(x)\rangle | x \in E\},\\
&@\{\langle x, [\inf M_A(x) + \alpha - \alpha \inf M_A(x), \sup M_A(x) + \beta - \beta \sup M_A(x)],\\
&[\gamma \inf N_A(x), \delta \sup N_A(x)]\rangle | x \in E\},\\
&= \Bigg\{\Bigg\langle x, \Bigg[\frac{\alpha \inf M_A(x) + \inf M_A(x) + \alpha - \alpha \inf M_A(x)}{2},\\
&\frac{\beta \sup M_A(x) + \sup M_A(x) + \beta - \beta \sup M_A(x)}{2}\Bigg],\\
&\Bigg[\frac{\inf N_A(x) + \gamma - \gamma \inf N_A(x) + \gamma \inf N_A(x)}{2},\\
&\frac{\sup N_A(x) + \delta - \delta \sup N_A(x) + \delta \sup N_A(x)}{2}\Bigg]\Bigg\rangle \Bigg| x \in E\Bigg\}\\
&= \Bigg\{\Bigg\langle x, \Bigg[\frac{\inf M_A(x) + \alpha}{2}, \frac{\sup M_A(x) + \beta}{2}\Bigg],\\
&\Bigg[\frac{\gamma + \gamma \inf N_A(x)}{2}, \frac{\delta + \delta \sup N_A(x)}{2}\Bigg]\Bigg\rangle \Bigg| x \in E\Bigg\}\\
&A @ \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\}.
\end{aligned}
$$

From the above discussion we see that the four new operators are simultaneously modal as well as level operators. By this reason, we legitimately can call them modal-level operators.

Finally, following the above notation, we can denote:

$$
A @ \{\langle x, [\alpha, \beta], [\gamma, \delta]\rangle | x \in E\} = O^{@}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A).
$$

4 Conclusion

In [1, 2, 3], to each of the 189 intuitionistic fuzzy implications introduced in [5], three different intuitionistic fuzzy conjunctions and disjunctions are juxtaposed. Therefore, they can be basis for introducing a lot of new O- (and therefore, P-, Q-, H- and J-) operators. Each of these new operators can be extended in the ways the standard modal and level operators are extended. Simultaneously, some of the operators, introduced for the IFS-case, have to be extended to IVIFS-form (e.g., the G. Çuvalcıoğlu's operators, [12]).

The applications of these operators will be interesting. For example, they can be used in different Data Mining processes as of decision making, of pattern recognition and a lot of others. One of the novel Data Mining tools is the IFS-based intercriteria analysis (see, e.g., [6, 11, 13]). It can be easily seen that the operators, discussed in the paper, can be used for modification of InterCriteria Analysis results and this will be an object of future research.

Acknowledgements

The present research has been supported by the Bulgarian National Science Fund under Grant Ref. No. KP-06-N-22/1 "Theoretical Research and Applications of InterCriteria Analysis".

References

[1] Angelova, N., & Stoenchev, M. (2015/2016). Intuitionistic fuzzy conjunctions and disjunctions from first type. *Annual of "Informatics" Section, Union of Scientists in Bulgaria*, 8, 1–17.

[2] Angelova, N., Stoenchev, M., & Todorov, V. (2017). Intuitionistic fuzzy conjunctions and disjunctions from second type. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, 13, 143–170.

[3] Angelova, N., & Stoenchev, M. (2017). Intuitionistic fuzzy conjunctions and disjunctions from third type. *Notes on Intuitionistic Fuzzy Sets*, 23 (5), 29—41.

[4] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer, Heidelberg.

[5] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin.

[6] Atanassov, K. (2014). *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham.

[7] Atanassov, K. (2018). On interval-valued intuitionistic fuzzy modal operators. *Notes on Intuitionistic Fuzzy Sets*, 24 (1), 1–12.

[8] Atanassov, K. (2018). Intuitionistic fuzzy modal operators of second type over interval-valued intuitionistic fuzzy sets. Part 1. *Notes on Intuitionistic Fuzzy Sets*, 2 (2), 8–17.

[9] Atanassov, K. (2018). Intuitionistic fuzzy modal operators of second type over interval-valued intuitionistic fuzzy sets. Part 2. *Notes on Intuitionistic Fuzzy Sets*, 24 (3), 1–10.

[10] Atanassov, K., & Gargov, G. (1989) Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31 (3), 343–349.

[11] Atanassov K., Mavrov, D., & Atanassova, V. (2014). Intercriteria Decision Making: A New Approach for Multicriteria Decision Making, Based on Index Matrices and Intuitionistic Fuzzy Sets. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, 11, 1—8.

[12] Çuvalcioğlu, G. (2007). Some properties of $E_{\alpha,\beta}$ operator. *Advanced Studies in Contemporary Mathematics*, 14 (2), 305–310.

[13] Traneva, V., & Tranev, S. (2017). *Index Matrices as a Tool for Managerial Decision Making*, Publ. House of the Union of Scientists in Bulgaria (in Bulgarian).