Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 23, 2017, No. 4, 91–105

Approximations of crisp set and intuitionistic fuzzy set based on intuitionistic fuzzy normal subgroup

Prasenjit Mandal[1] and A. S. Ranadive[2]

[1] Bhalukdungri Jr High School
Raigara, Purulia (W.B.), 723153, India
e-mail: prasenjitmandal08@yahoo.com

[2] Department of Pure and Applied Mathematics, Guru Ghasidas University
Bilaspur (C.G.), India
e-mail: asranadive04@yahoo.co.in

Received: 14 March 2017 **Accepted:** 1 November 2017

Abstract: We consider a group G, with identity element e, as a universal set and assume that the knowledge about objects is restricted by an intuitionistic fuzzy normal subgroup $A = (\mu_A, \nu_A)$ of G. For each $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, the set $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$ is a congruence relation on G, where $U(\mu_A, \alpha) = \{(x, y) \in G \times G : \mu_A(xy^{-1}) \geq \alpha\}$ and $U(\nu_A, \beta) = \{(x, y) \in G \times G : \nu_A(xy^{-1}) \leq \beta\}$. In this paper, the notion of $U(A, \alpha, \beta)$-lower and $U(A, \alpha, \beta)$-upper approximation of a non-empty subset of G and an intuitionistic fuzzy set of G are introduced and some important properties of the above approximations are presented.
Keywords: Rough set, Fuzzy set, Intuitionistic fuzzy set, Intuitionistic fuzzy normal subgroup, Congruence relation, Lower and Upper approximations of crisp and fuzzy sets.
AMS Classification: 03E72, 06F35, 08A72, 03E99.

1 Introduction

After the introduction of the concept of fuzzy sets by Zadeh [18] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2] as a generalization of the notion of the fuzzy set. This subject has been studied by [4, 7, 12] and others.

In 1982, Pawlak [16] proposed rough set theory as a new mathematical tool for dealing with uncertainties, which is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. The notion of rough set and its properties were applied to various areas such as semigroups [15], groups [5, 13, 14], rings [6], ideals [8, 17], modules [9], lattices [11] and fuzzy sets [10].

In this paper, we are to study the rough set based on intuitionistic fuzzy normal subgroup. In fact, the study concerns the relationship between rough sets, intuitionistic fuzzy sets and group theory. First of all, in Section 2, we review some basic definitions and notation. In Section 3 & 4, we consider a group as a universe set and we assume that the knowledge about objects is restricted by intuitionistic fuzzy normal subgroups. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy normal subgroup of G and $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. For a $b \in G$, we say that a is congruent to $b \pmod{A} = (\mu_A, \nu_A)$, written $a \equiv_{(\alpha,\beta)} b$ if $\mu_A(xy^{-1}) \geq \alpha$ and $\nu_A(xy^{-1}) \leq \beta$. In Section 2.4, we describe some properties of this congruence relation. These properties are used in Section 3 & 4. In Section 3, we analyzed the lower and upper approximations of a subset of a group with respect to the above congruence relation. In Section 4, we analyzed the lower and upper approximations of an intuitionistic fuzzy set of a group with respect to the above congruence relation, which is the main aim of this paper. At last, a conclusion is presented.

2 Definitions and notation

We introduce in this section some definitions and notation used in the present paper.

2.1 Rough set

Let U be a universal set. For an equivalence relation θ on U, the set of the elements of U that are related to $x \in U$, is called the equivalence class of x, and is denoted by $[x]_\theta$. A pair (U, θ), where $U \neq \emptyset$ and θ is an equivalence relation of U, is called an approximation space.

Definition 1 ([8]). *For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $(U, \theta, ...) : P(U) \to P(U) \times P(U)$ define for every $X \in P(U)$ by $(U, \theta, X) = ((\underline{U}, \theta, X), (\overline{U}, \theta, X))$, where $(\underline{U}, \theta, X) = \{x \in U : [x]_\theta \subseteq X\}$, $(\overline{U}, \theta, X) = \{x \in U : [x]_\theta \cap X \neq \emptyset\}$. $(\underline{U}, \theta, X)$ is called a lower rough approximation of X in (U, θ), where as $(\overline{U}, \theta, X)$ is called a upper rough approximation of X in (U, θ).*

2.2 Intuitionistic fuzzy set

Throughout this paper G is a group. By a fuzzy set of G is a function $\mu : G \to [0, 1]$. Let μ be a fuzzy set of G. For each $t \in [0, 1]$ $\mu_t = \{x \in G : \mu(x) \geq t\}$ is called a t-level set of μ or t-cut of μ. Let μ and λ be two fuzzy sets of G. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \leq \mu(x)$ for all $x \in G$, and $\mu \cap \lambda$ is defined by $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$ for all $x \in G$ and $(\mu * \lambda)(x) = \vee_{yz=x}(\mu(y) \wedge \lambda(z))$.

Definition 2 ([1]). *An intuitionistic fuzzy set (IFS) A in a non-empty set G is an object having the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$$

where the functions $\mu_A : G \to [0,1]$ *and* $\nu_A : G \to [0,1]$ *defined the degree of membership and the degree of membership and the degree of non-membership of the element* $x \in G$ *to the set A, which is a subset of G, respectively, and*

$$0 \leq \mu_A(x) \leq \nu_A(x) \leq 1$$

for all $x \in G$*. An IFS* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$ *in G can be identified to an ordered pair* (μ_A, ν_A) *in* $[0,1]^G \times [0,1]^G$*. For the sake of simplicity, we shall use the symbol* $A = (\mu_A, \nu_A)$ *for the IFS* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$*.*

Definition 3 ([1]). *Let* $A = (\mu_A, \nu_A)$ *be an IFS of G and* $\alpha, \beta \in [0,1]$ *such that* $\alpha + \beta \leq 1$*. The sets* $A_\alpha = \{x \in G : \mu_A(x) \geq \alpha\}$ *and* $A_\beta = \{x \in G : \nu_A(x) \leq \beta\}$ *are called, respectively, the* α*-level set or* α*-cut of* μ_A *and* β*-level set or* β*-cut of* ν_A*; and the set* $A_{(\alpha,\beta)} = A_\alpha \cap A_\beta$ *is called a* (α, β)*-level set or* (α, β)*-cut of the IFS* $A = (\mu_A, \nu_A)$*.*

Definition 4 ([1]). *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be any two IFSs of G, then*

(i) $A \subseteq B \Leftrightarrow (\forall x \in G)(\mu_A(x) \leq \mu_B(x),\ \nu_A(x) \geq \nu_B(x))$.

(ii) $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$.

(iii) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B) = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle | x \in\}$.

(iv) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle | x \in G\}$.

According to [3] we have the following definition.

Definition 5. *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be any two IFSs of G. Then we defined the IFS* $A * B = (\mu_{A*B}, \nu_{A*B})$ *of G is given by*

$$\mu_{A*B}(x) = \vee_{yz=x}(\mu_A(y) \wedge \mu_B(z))\ \forall y, z \in G, yz = x,$$
$$\nu_{A*B}(x) = \wedge_{yz=x}(\nu_A(y) \vee \nu_B(z))\ \forall y, z \in G, yz = x.$$

2.3 Intuitionistic fuzzy normal subgroup

A fuzzy set μ of G is called a fuzzy subgroup of G if it has the following properties:

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$,

(ii) $\mu(x^{-1}) = \mu(x)$ for all $x \in G$.

A fuzzy subgroup μ of G is called normal subgroup of G if $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

For a normal fuzzy subgroup μ of G, we have the following:

(i) $\mu(x) \leq \mu(e)$ for all $x \in G$.

(ii) $\mu(xy^{-1}) = \mu(e)$ implies $\mu(x) = \mu(y)$, where $x, y \in G$.

(iii) If μ and λ be a fuzzy normal subgroup of G, then so is $\mu \cap \lambda$.

(iv) A fuzzy set μ of G is a fuzzy normal subgroup of G if and only if for any $t \in [0,1]$, such that $\mu_t \neq \emptyset$, μ_t is a normal subgroup of G.

Definition 6 ([12]). *An IFS $A = (\mu_A, \nu_A)$ of a group G is called an intuitionistic fuzzy subgroup (IFSG) of G if it has the following properties:*

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ for all $x, y \in G$,

(ii) $\mu_A(x^{-1}) = \mu_A(x)$ for all $x \in G$,

(iii) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for all $x, y \in G$,

(iv) $\nu_A(x^{-1}) = \nu_A(x)$ for all $x \in G$,

An IFSG $A = (\mu_A, \nu_A)$ of G is called an intuitionistic fuzzy normal subgroup (IFNSG) of G if $\mu_A(xy) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(yx)$ for all $x, y \in G$.

Lemma 1. For an IFNSG $A = (\mu_A, \nu_A)$ of G, we have the following:

(i) $\mu_A(x) \leq \mu_A(e)$ and $\nu_A(x) \geq \nu_A(e)$ for all $x \in G$,

(ii) $\mu_A(xy^{-1}) = \mu_A(e)$ and $\nu_A(xy^{-1}) = \nu_A(e)$ implies $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$ where $x, y \in G$,

(iii) If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two IFNSGs of G, then so is $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(iv) An IFS $A = (\mu_A, \nu_A)$ of G is an IFNSG of G if and only if $A_{(\alpha,\beta)}$ is a normal subgroup of G, for any $\alpha, \beta \in [0,1]$ $(\alpha + \beta \leq 1)$ and $A_{(\alpha,\beta)} \neq \emptyset$.

2.4 Congruence relation

Let μ be a normal subgroup of G. For each $t \in [0,1]$, the set $U(\mu, t) = \{(a,b) \in G \times G : \mu(ab^{-1}) \geq t\}$ is called a t-level relation of μ. It is no difficult to show that $U(\mu, t)$ is a congruence relation.

Definition 7. *Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. The sets $U(\mu_A, \alpha) = \{(a,b) \in G \times G : \mu_A(ab^{-1}) \geq \alpha\}$ and $U(\nu_A, \beta) = \{(a,b) \in G \times G : \nu_A(ab^{-1}) \leq \beta\}$ are called, respectively, the α-level relation of μ_A and β-level relation of ν_A; and the set $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$ is called a (α, β)-level relation of $A = (\mu_A, \nu_A)$.*

The following two lemmas are straightforward.

Lemma 2. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. Then $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$ is a congruence relation of G.

Lemma 3. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. Then

(i) $U(\mu_A \wedge \mu_B, \alpha) = U(\mu_A, \alpha) \cap U(\mu_B, \alpha)$.

(ii) $U(\nu_A \vee \nu_B, \beta) = U(\nu_A, \beta) \cup U(\nu_B, \beta)$.

We denote $[x]_{(A,\alpha,\beta)} = [x]_{(\mu_A,\alpha)} \cap [x]_{(\nu_A,\beta)}$ the equivalence class of $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$.

Lemma 4. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. Then

(i) $[a]_{(\mu_A,\alpha)}[b]_{(\mu_A,\alpha)} = [ab]_{(\mu_A,\alpha)}$ and $[a]_{(\nu_A,\beta)}[b]_{(\nu_A,\beta)} = [ab]_{(\nu_A,\beta)}$.

(ii) $[a^{-1}]_{(\mu_A,\alpha)} = ([a]_{(\mu_A,\alpha)})^{-1}$ and $[a^{-1}]_{(\nu_A,\beta)} = ([a]_{(\nu_A,\beta)})^{-1}$.

Proof. (i) Let $x \in [a]_{(\mu_A,\alpha)}[b]_{(\mu_A,\alpha)}$. Then there exists $y \in [a]_{(\mu_A,\alpha)}$ and $z \in [b]_{(\mu_A,\alpha)}$ such that $yz = x$. Since $(a, y) \in U(\mu_A, \alpha)$ and $(b, z) \in U(\mu_A, \alpha)$. By Lemma 2, we have $(ab, yz) \in U(\mu_A, \alpha)$ or $(ab, x) \in U(\mu_A, \alpha)$, and so $x \in [ab]_{(\mu_A,\alpha)}$.

Conversely, let $x \in [ab]_{(\mu_A,\alpha)}$, then $(x, ab) \in U(\mu_A, \alpha)$. Hence $(xb^{-1}, a) \in U(\mu_A, \alpha)$ and so $xb^{-1} \in [a]_{(\mu_A,\alpha)}$ i.e., $x \in [a]_{(\mu_A,\alpha)}b$, which implies $x \in [a]_{(\mu_A,\alpha)}[b]_{(\mu_A,\alpha)}$.

The proof of the second part is similar to the proof of first part.

(ii) We have

$$\begin{aligned} x \in [a^{-1}]_{(\mu_A,\alpha)} &\Leftrightarrow (x, a^{-1}) \in U(\mu_A, \alpha) \\ &\Leftrightarrow (e, a^{-1}x^{-1}) \in U(\mu_A, \alpha) \\ &\Leftrightarrow (a, x^{-1}) \in U(\mu_A, \alpha) \\ &\Leftrightarrow x^{-1} \in [a]_{(\mu_A,\alpha)} \\ &\Leftrightarrow x \in ([a]_{(\mu_A,\alpha)})^{-1}. \end{aligned}$$

The rest is similar. □

Lemma 5. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. For any $a \in G$, we have $a[e]_{(\mu_A,\alpha)} = [a]_{(\mu_A,\alpha)}$ and $a[e]_{(\nu_A,\beta)} = [a]_{(\nu_A,\beta)}$.

Proof. Let $a \in G$, and then we have

$$x \in a[e]_{(\mu_A,\alpha)} \Leftrightarrow a^{-1}x \in [e]_{(\mu_A,\alpha)} \Leftrightarrow (a^{-1}x, e) \in U(\mu_A, \alpha) \Leftrightarrow (x, a) \in U(\mu_A, \alpha) \Leftrightarrow x \in [a]_{(\mu_A,\alpha)}.$$

The rest is similar. □

Lemma 6. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. If $A \subseteq B$, then $[x]_{(\mu_A,\alpha)} \subseteq [x]_{(\mu_B,\alpha)}$ and $[x]_{(\nu_A,\beta)} \subseteq [x]_{(\nu_B,\beta)}$.

Proof. We have $y \in [x]_{(\mu_A,\alpha)} \Rightarrow (x,y) \in U(\mu_A,\alpha) \Rightarrow \mu_A(xy^{-1}) \geq \alpha \Rightarrow \mu_B(xy^{-1}) \geq \alpha \Rightarrow (x,y) \in U(\mu_B,\alpha) \Rightarrow y \in [x]_{(\mu_B,\alpha)}$. The rest is similar.

Let $A = (\mu_A,\nu_A)$ and $B = (\mu_B,\nu_B)$ be two IFNSGs of G and $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \leq 1$. The composition of congruence relations $U(A,\alpha,\beta) = U(\mu_A,\alpha) \cap U(\nu_A,\beta)$ and $U(B,\alpha,\beta) = U(\mu_B,\alpha) \cap U(\nu_B,\beta)$ is defined as follows:

$$U(A,\alpha,\beta) \circ U(B,\alpha,\beta) = U(\mu_{A\circ B},\alpha) \cap U(\nu_{A\circ B},\beta),$$

where $U(\mu_{A\circ B},\alpha) = \{(a,b) \in G \times G : (a,c) \in U(\mu_A,\alpha), (c,b) \in U(\mu_B,\alpha)\}$ and $U(\nu_{A\circ B},\beta) = \{(a,b) \in G \times G : (a,c) \in U(\nu_A,\beta), (c,b) \in U(\nu_B,\beta)\}$. It is no difficult to show that $U(\mu_{A\circ B},\alpha)$ and $U(\nu_{A\circ B},\beta)$ are also congruence relations. Therefore, $U(A,\alpha,\beta) \circ U(B,\alpha,\beta)$ is a congruence relation. We denote this congruence relation by $U(A\circ B,\alpha,\beta) = U(\mu_{A\circ B},\alpha) \cap U(\nu_{A\circ B},\beta)$. □

Lemma 7. Let $A = (\mu_A,\nu_A)$ and $B = (\mu_B,\nu_B)$ be two IFNSGs of G and $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \leq 1$. Then $U(\mu_{A\circ B},\alpha) \subseteq U(\mu_{A*B},\alpha)$ and $U(\nu_{A\circ B},\beta) \subseteq U(\nu_{A*B},\beta)$.

Proof. Let $(a,b) \in U(\mu_{A\circ B},\alpha)$. Then there exists $c \in G$ such that $(a,c) \in U(\mu_A,\alpha)$ and $(c,b) \in U(\mu_B,\alpha)$. Therefore we have $\mu_A(ac^{-1}) \geq \alpha$ and $\mu_A(cb^{-1}) \geq \alpha$. Then $\mu_{A*B}(ab^{-1}) = \vee_{yz=ab^{-1}}(\mu_A(y) \wedge \mu_B(z)) \geq \mu_A(ac^{-1}) \wedge \mu_A(cb^{-1}) \geq \alpha$ and so $(a,b) \in U(\mu_{A*B},\alpha)$. The rest is similar. □

Lemma 8. Let $A = (\mu_A,\nu_A)$ and $B = (\mu_B,\nu_B)$ be two IFNSGs of G and $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \leq 1$ with finite image. Then $U(\mu_{A\circ B},\alpha) = U(\mu_{A*B},\alpha)$ and $U(\nu_{A\circ B},\beta) = U(\nu_{A*B},\beta)$.

Proof. We are only to proof second part. By Lemma 7, we have $U(\nu_{A\circ B},\beta) \subseteq U(\nu_{A*B},\beta)$, therefore we can show that $U(\nu_{A\circ B},\beta) \supseteq U(\nu_{A*B},\beta)$. For this let $(x,y) \in U(\nu_{A*B},\beta)$, then $\nu_{A*B}(xy^{-1}) \leq \beta$. We have $\wedge_{ab=xy^{-1}}(\nu_A(a) \vee \nu_B(b)) \leq \beta$. Since $Im(\nu_A)$ and $Im(\nu_B)$ are finite, then $\nu_A(a_0) \vee \nu_B(b_0) \leq \beta$ for some $a_0, b_0 \in G$, such that $xy^{-1} = a_0 b_0$. Thus $\nu_A(a_0) \leq \beta$ or $\nu_B(b_0) \leq \beta$. Now we have $\nu_A(a_0 e^{-1}) \leq \beta$ or $\nu_B(a_0^{-1}xy^{-1}) \leq \beta$ i.e., we have $\nu_A(a_0 e^{-1}) \leq \beta$ or $\nu_B(xy^{-1}a_0^{-1}) \leq \beta$, which imply $(a_0,e) \in U(\nu_B,\beta)$ or $(xy^{-1},a_0) \in U(\nu_B,\beta)$. Therefore $(xy^{-1},e) \in U(\nu_{A\circ B},\beta)$ i.e., $(x,y) \in U(\nu_{A\circ B},\beta)$. □

3 Approximations of crisp set based on IFNSGs

Let $A = (\mu_A,\nu_A)$ and $B = (\mu_B,\nu_B)$ be two IFNSGs of G and $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \leq 1$. By Lemma 2, $U(A,\alpha,\beta) = U(\mu_A,\alpha) \cap U(\nu_A,\beta)$ is a congruence relation on G. Therefore, when $U = G$ and $\theta = U(A,\alpha,\beta)$, then we use $(G,A,\alpha,\beta) = ((G,\mu_A,\alpha),(G,\nu_A,\beta))$ instead of approximation space (U,θ). In the rest of this paper $A = (\mu_A,\nu_A)$ is an IFNSG of G.

Definition 8. *Let $U(A,\alpha,\beta) = U(\mu_A,\alpha) \cap U(\nu_A,\beta)$ be a congruence relation of an IFNSG of G and $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \leq 1$. Let X be a non-empty subset of G. Then we defined the sets $\underline{U}(\mu_A,\alpha,X)$, $\overline{U}(\mu_A,\alpha,X)$, $\underline{U}(\nu_A,\beta,X)$ and $\overline{U}(\nu_A,\beta,X)$ as follows:*

$$\underline{U}(\mu_A,\alpha,X) = \{x \in G : [x]_{(\mu_A,\alpha)} \subseteq X\} \text{ and } \overline{U}(\mu_A,\alpha,X) = \{x \in G : [x]_{(\mu_A,\alpha)} \cap X \neq \emptyset\};$$
$$\underline{U}(\nu_A,\beta,X) = \{x \in G : [x]_{(\mu_A,\beta)} \cap X \neq \emptyset\} \text{ and } \overline{U}(\nu_A,\beta,X) = \{x \in G : [x]_{(\mu_A,\beta)} \subseteq X\}.$$

The sets $\underline{U}(\mu_A, \alpha, X)$ and $\overline{U}(\mu_A, \alpha, X)$ are called, respectively, the lower and upper approximations of the set X with respect to $U(\mu_A, \alpha)$; and the sets $\underline{U}(\nu_A, \beta, X)$ and $\overline{U}(\nu_A, \beta, X)$ are called, respectively, the lower and upper approximations of the set X with respect to $U(\nu_A, \beta)$; and the sets $\underline{U}(A, \alpha, \beta, X) = ((\underline{U}(\mu_A, \alpha, X), \underline{U}(\nu_A, \beta, X))$ and $\overline{U}(A, \alpha, \beta, X) = (\overline{U}(\mu_A, \alpha, X), \overline{U}(\nu_A, \beta, X))$ are called, respectively, the lower and upper approximations of the set X with respect to $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$.

$U(A, \alpha, \beta, X) = (\underline{U}(A, \alpha, \beta, X), \overline{U}(A, \alpha, \beta, X))$ is called a rough set with respect to $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$.

The following proposition is exactly obtained from Definition 8.

Proposition 1. For every approximation space $(G, A, \alpha, \beta) = ((G, \mu_A, \alpha), (G, \nu_A, \beta))$ and every subsets X, Y of G we have:

(1) $\underline{U}(\mu_A, \alpha, X) \subseteq X \subseteq \overline{U}(\mu_A, \alpha, X)$ and $\underline{U}(\nu_A, \beta, X) \supseteq X \supseteq \overline{U}(\nu_A, \beta, X)$.

(2) $\underline{U}(\mu_A, \alpha, \emptyset) = \emptyset = \overline{U}(\mu_A, \alpha, \emptyset)$ and $\underline{U}(\nu_A, \beta, \emptyset) = \emptyset = \overline{U}(\nu_A, \beta, \emptyset)$.

(3) $\underline{U}(\mu_A, \alpha, G) = G = \overline{U}(\mu_A, \alpha, G)$ and $\underline{U}(\nu_A, \beta, G) = G = \overline{U}(\nu_A, \beta, G)$.

(4) If $X \subseteq Y$, then

$$\underline{U}(\mu_A, \alpha, X) \subseteq \underline{U}(\mu_A, \alpha, Y) \text{ and } \underline{U}(\nu_A, \beta, X) \subseteq \underline{U}(\nu_A, \beta, Y),$$
$$\overline{U}(\mu_A, \alpha, X) \subseteq \overline{U}(\mu_A, \alpha, Y) \text{ and } \overline{U}(\nu_A, \beta, X) \subseteq \overline{U}(\nu_A, \beta, Y).$$

(5) $\underline{U}(\mu_A, \alpha, (\underline{U}(\mu_A, \alpha, X)) = \underline{U}(\mu_A, \alpha, X)$ and $\underline{U}(\nu_A, \beta, (\underline{U}(\nu_A, \beta, X)) = \underline{U}(\nu_A, \beta, X)$.

(6) $\overline{U}(\mu_A, \alpha, (\overline{U}(\mu_A, \alpha, X)) = \overline{U}(\mu_A, \alpha, X)$ and $\overline{U}(\nu_A, \beta, (\overline{U}(\nu_A, \beta, X)) = \overline{U}(\nu_A, \beta, X)$.

(7) $\overline{U}(\mu_A, \alpha, (\underline{U}(\mu_A, \alpha, X)) = \underline{U}(\mu_A, \alpha, X)$ and $\overline{U}(\nu_A, \beta, (\underline{U}(\nu_A, \beta, X)) = \underline{U}(\nu_A, \beta, X)$.

(8) $\underline{U}(\mu_A, \alpha, (\overline{U}(\mu_A, \alpha, X)) = \overline{U}(\mu_A, \alpha, X)$ and $\underline{U}(\nu_A, \beta, (\overline{U}(\nu_A, \beta, X)) = \overline{U}(\nu_A, \beta, X)$.

(9) $\underline{U}(\mu_A, \alpha, X \cap Y) = \underline{U}(\mu_A, \alpha, X) \cap \underline{U}(\mu_A, \alpha, Y)$ and $\underline{U}(\nu_A, \beta, X \cap Y) \subseteq \underline{U}(\nu_A, \beta, X) \cap \underline{U}(\nu_A, \beta, Y)$.

(10) $\overline{U}(\mu_A, \alpha, X \cap Y) \subseteq \overline{U}(\mu_A, \alpha, X) \cap \overline{U}(\mu_A, \alpha, Y)$ and $\overline{U}(\nu_A, \beta, X \cap Y) = \overline{U}(\nu_A, \beta, X) \cap \overline{U}(\nu_A, \beta, Y)$.

(11) $\underline{U}(\mu_A, \alpha, X \cup Y) \supseteq \underline{U}(\mu_A, \alpha, X) \cup \underline{U}(\mu_A, \alpha, Y)$ and $\underline{U}(\nu_A, \beta, X \cup Y) = \underline{U}(\nu_A, \beta, X) \cup \underline{U}(\nu_A, \beta, Y)$.

(12) $\overline{U}(\mu_A, \alpha, X \cup Y) = \overline{U}(\mu_A, \alpha, X) \cup \overline{U}(\mu_A, \alpha, Y)$ and $\overline{U}(\nu_A, \beta, X \cup Y) \supseteq \overline{U}(\nu_A, \beta, X) \cup \overline{U}(\nu_A, \beta, Y)$.

(13) $\underline{U}(\mu_A, \alpha, [x]_{(\mu_A, \alpha)}) = \overline{U}(\mu_A, \alpha, [x]_{(\mu_A, \alpha)})$ and $\underline{U}(\nu_A, \beta, [x]_{(\nu_A, \beta)}) = \overline{U}(\nu_A, \beta, [x]_{(\nu_A, \beta)})$, for all $x \in G$.

Proposition 2. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G, then

(i) $\overline{U}(\mu_A \wedge \mu_B, \alpha, X) \subseteq \overline{U}(\mu_A, \alpha, X) \cap \overline{U}(\mu_B, \alpha, X)$.

(ii) $\underline{U}(\nu_A \wedge \nu_B, \beta, X) \subseteq \underline{U}(\nu_A, \beta, X) \cap \underline{U}(\nu_B, \beta, X)$.

Proof. We have

$$\begin{aligned} x \in \overline{U}(\mu_A \wedge \mu_B, \alpha, X) &\Rightarrow [x]_{(\mu_A \wedge \mu_B, \alpha)} \cap X \neq \emptyset \\ &\Rightarrow \exists a \in [x]_{(\mu_A \wedge \mu_B, \alpha)} \cap X \\ &\Rightarrow (a, x) \in U(\mu_A \wedge \mu_B, \alpha) \text{ and } a \in X \\ &\Rightarrow (\mu_A \wedge \mu_B)(ax^{-1}) \geq \alpha \text{ and } a \in X \\ &\Rightarrow \mu_A(ax^{-1}) \geq \alpha \text{ and } \mu_B(ax^{-1}) \geq \alpha \text{ and } a \in X \\ &\Rightarrow (a, x) \in U(\mu_A, \alpha), a \in X \text{ and } (a, x) \in U(\mu_B, \alpha), a \in X \\ &\Rightarrow [x]_{(\mu_A, \alpha)} \cap X \neq \emptyset \text{ and } [x]_{(\mu_B, \alpha)} \cap X \neq \emptyset \\ &\Rightarrow x \in \overline{U}(\mu_A, \alpha, X) \text{ and } x \in \overline{U}(\mu_A, \alpha, X) \end{aligned}$$

Therefore, $\overline{U}(\mu_A \wedge \mu_B, \alpha, X) \subseteq \overline{U}(\mu_A, \alpha, X) \cap \overline{U}(\mu_B, \alpha, X)$. The rest is similar. □

Proposition 3. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G, then

(i) $\underline{U}(\mu_A, \alpha, X) \cap \underline{U}(\mu_B, \alpha, X) \subseteq \underline{U}(\mu_A \wedge \mu_B, \alpha, X)$.

(ii) $\overline{U}(\nu_A, \beta, X) \cap \overline{U}(\nu_B, \beta, X) \subseteq \overline{U}(\nu_A \wedge \nu_B, \beta, X)$.

Proof. We have

$$\begin{aligned} x \in \underline{U}(\mu_A, \alpha, X) \cap \underline{U}(\mu_B, \alpha, X) &\Rightarrow x \in \underline{U}(\mu_A, \alpha, X) \text{ and } x \in \underline{U}(\mu_B, \alpha, X) \\ &\Rightarrow [x]_{(\mu_A, \alpha)} \subseteq X \text{ and } [x]_{(\mu_B, \alpha)} \\ &\Rightarrow [x]_{(\mu_A \wedge \mu_B, \alpha)} \subseteq X \;\text{ By Lemma 6} \\ &\Rightarrow x \in \underline{U}(\mu_A \wedge \mu_B, \alpha, X). \end{aligned}$$

Therefore, $\underline{U}(\mu_A, \alpha, X) \cap \underline{U}(\mu_B, \alpha, X) \subseteq \underline{U}(\mu_A \wedge \mu_B, \alpha, X)$. The rest is similar. □

Lemma 9. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If H is a normal subgroup of G, then $\overline{U}(\mu_A, \alpha, H)$ and $\underline{U}(\nu_A, \beta, H)$ are normal subgroups of G.

Proof. Let $a, b \in \overline{U}(\mu_A, \alpha, H)$ and $g \in G$, then $[a]_{(\mu_A, \alpha)} \cap H \neq \emptyset$ and $[b]_{(\mu_A, \alpha)} \cap H \neq \emptyset$. So there exists $x \in [a]_{(\mu_A, \alpha)} \cap H$ and $y \in [b]_{(\mu_A, \alpha)} \cap H$. Since H is a normal subgroup of G, we have $xy^{-1} \in H$ and $gxg^{-1} \in H$, where $g \in G$. By Lemma 4, we have $xy^{-1} \in [a]_{(\mu_A, \alpha)}([b]_{(\mu_A, \alpha)})^{-1} = [a]_{(\mu_A, \alpha)}[b^{-1}]_{(\mu_A, \alpha)} = [ab^{-1}]_{(\mu_A, \alpha)}$. Hence, $xy^{-1} \in [ab^{-1}]_{(\mu_A, \alpha)} \cap H$ i.e., $[ab^{-1}]_{(\mu_A, \alpha)} \cap H \neq \emptyset$, which implies that $ab^{-1} \in \overline{U}(\mu_A, \alpha, H)$.

Since $(x, a) \in U(\mu_A, \alpha)$, then $\mu_A(xa^{-1}) \geq \alpha$. Now we have $\mu_A(gxg^{-1}(gag^{-1})^{-1}) = \mu_A(gxg^{-1}ga^{-1}g^{-1}) = \mu_A(gxa^{-1}g^{-1}) = \mu_A(xa^{-1}) \geq \alpha$. Hence, $(gxg^{-1}, gag^{-1}) \in U(\mu_A, \alpha)$, i.e., $gxg^{-1} \in [gag^{-1}]_{(\mu_A,\alpha)}$, thus $gxg^{-1} \in [gag^{-1}]_{(\mu_A,\alpha)} \cap H$, which implies $[gag^{-1}]_{(\mu_A,\alpha)} \cap H \neq \emptyset$. Therefore, $gag^{-1} \in \overline{U}(\mu_A, \alpha, H)$. Hence $\overline{U}(\mu_A, \alpha, H)$ is a normal subgroup of G. The rest is similar. □

Lemma 10. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. Let X be a non-empty subset of G. If $\underline{U}(\mu_A, \alpha, X)$ and $\overline{U}(\nu_A, \beta, X)$ are non-empty sets then, $[e]_{(\mu_A,\alpha)} \subseteq X$ and $[e]_{(\nu_A,\beta)} \subseteq X$.

Proof. Since $\underline{U}(\mu_A, \alpha, X) \neq \emptyset$, then there exists $x \in \underline{U}(\mu_A, \alpha, X)$, i.e., $[x]_{(\mu_A,\alpha)} \subseteq X$. So $([x]_{(\mu_A,\alpha)})^{-1} \subseteq (X)^{-1} \subseteq \{a^{-1} : a \in X\} = X$. Now by Lemma 4 we have $[e]_{(\mu_A,\alpha)} = [xx^{-1}]_{(\mu_A,\alpha)} = [x]_{(\mu_A,\alpha)}[x^{-1}]_{(\mu_A,\alpha)} = [x]_{(\mu_A,\alpha)}([e]_{(\mu_A,\alpha)})^{-1} \subseteq XX \subseteq X$. The rest is similar. □

Lemma 11. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If H is a normal subgroup of G, then $\underline{U}(\mu_A, \alpha, H)$ and $\overline{U}(\nu_A, \beta, H)$ are non-empty sets, then its equal to H.

Proof. By Proposition 1(1), we have $\underline{U}(\mu_A, \alpha, H) \subseteq H$. We show that $H \subseteq \underline{U}(\mu_A, \alpha, H)$. For this, let $h \in H$. By Lemma 10, we have $[e]_{(\mu_A,\alpha)} \subseteq H$. Since H is a normal subgroup of G, we have $h[e]_{(\mu_A,\alpha)} \subseteq hH \subseteq H$. By Lemma 5, we have $[h]_{(\mu_A,\alpha)} \subseteq H$, which implies that $h \in \underline{U}(\mu_A, \alpha, H)$. Therefore, $H \subseteq \underline{U}(\mu_A, \alpha, H)$. The rest is similar.

From Lemmas 9 and 11, the following Corollaries 1 and 2 are true. □

Corollary 1. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If H is a normal subgroup of G, then $\overline{U}(A, \alpha, \beta, H) = (\overline{U}(\mu_A, \alpha, H), \overline{U}(\nu_A, \beta, H))$ is a normal subgroup of G.

Corollary 2. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If H is a normal subgroup of G, then $\underline{U}(A, \alpha, \beta, H) = (\underline{U}(\mu_A, \alpha, H), \underline{U}(\nu_A, \beta, H))$ is a normal subgroup of G.

Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$; and $(\underline{U}(A, \alpha, \beta, H), \overline{U}(A, \alpha, \beta, H))$ a rough set in the approximation space (G, A, α, β). If $\underline{U}(A, \alpha, \beta, H) = (\underline{U}(\mu_A, \alpha, H), \underline{U}(\nu_A, \beta, H))$ and $\overline{U}(A, \alpha, \beta, H) = (\overline{U}(\mu_A, \alpha, H), \overline{U}(\nu_A, \beta, H))$ are normal subgroup of G, then we call $(\underline{U}(A, \alpha, \beta, H), \overline{U}(A, \alpha, \beta, H))$ a rough normal subgroup. Therefore, from Corollaries 1 and 2, we give the following Corollary.

Corollary 3. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If H is a normal subgroup of G, then $(\underline{U}(A, \alpha, \beta, H), \overline{U}(A, \alpha, \beta, H))$ a rough normal subgroup of G.

Proposition 4. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G and $A \subseteq B$, then

(i) $\overline{U}(\mu_A, \alpha, X) \subseteq \overline{U}(\mu_B, \alpha, X)$ and $\overline{U}(\nu_A, \beta, X) \subseteq \overline{U}(\nu_B, \beta, X)$.

(ii) $\underline{U}(\mu_B, \alpha, X) \subseteq \underline{U}(\mu_A, \alpha, X)$ and $\underline{U}(\nu_B, \beta, X) \subseteq \underline{U}(\nu_A, \beta, X)$.

Proof. (i) Let $x \in \overline{U}(\mu_A, \alpha, X)$, then $[x]_{(\mu_A,\alpha)} \cap X \neq \emptyset$. By Lemma 6, since $[x]_{(\mu_A,\alpha)} \subseteq [x]_{(\mu_B,\alpha)}$, we have $[x]_{(\mu_B,\alpha)} \cap X \neq \emptyset$, which implies that $x \in \overline{U}(\mu_B, \alpha, X)$. Again let, $y \in \overline{U}(\nu_B, \beta, X)$, then $[y]_{(\nu_A,\beta)} \subseteq X$. Now by Lemma 6, since $[y]_{(\nu_B,\beta)} \subseteq [y]_{(\nu_A,\beta)}$ we have $[y]_{(\nu_B,\beta)} \subseteq X$, which

implies $y \in \overline{U}(\nu_B, \beta, X)$.
(ii) The proof is similar to the proof of statement (i). $\square$

Proposition 5. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G and $U(A, \alpha, \beta) \subseteq U(B, \alpha, \beta)$ then,

(i) $\overline{U}(\mu_A, \alpha, X) \subseteq \overline{U}(\mu_B, \alpha, X)$ and $\overline{U}(\nu_A, \beta, X) \subseteq \overline{U}(\nu_B, \beta, X)$.

(ii) $\underline{U}(\mu_B, \alpha, X) \subseteq \underline{U}(\mu_A, \alpha, X)$ and $\underline{U}(\nu_B, \beta, X) \subseteq \underline{U}(\nu_A, \beta, X)$.

Proof. (i) Let $x \in \overline{U}(\mu_A, \alpha, X)$, then we have there exists $a \in [x]_{(\mu_A,\alpha)} \cap X$. Then $(a, x) \in U(\mu_A, \alpha) \subseteq (\mu_B, \alpha)$ and $a \in X$. Therefore, $a \in [x]_{(\mu_B,\alpha)} \cap X$, and so $x \in \overline{U}(\mu_B, \alpha, X)$. Again let, $y \in \overline{U}(\nu_B, \beta, X)$, then we have $[y]_{(\nu_A,\beta)} \subseteq X$. Since $[y]_{(\nu_B,\beta)} \subseteq [y]_{(\nu_A,\beta)}$, i.e., $[y]_{(\nu_B,\beta)} \subseteq X$, which implies $y \in \overline{U}(\nu_B, \beta, X)$.
(ii) The proof is similar to the proof of statement (i). $\square$

Proposition 6. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G, then

(i) $\overline{U}(\mu_{A\circ B}, \alpha, X) \subseteq \overline{U}(\mu_{A*B}, \alpha, X)$ and $\overline{U}(\nu_{A\circ B}, \beta, X) \subseteq \overline{U}(\nu_{A*B}, \beta, X)$.

(ii) $\underline{U}(\mu_{A*B}, \alpha, X) \subseteq \underline{U}(\mu_{A\circ B}, \alpha, X)$ and $\underline{U}(\nu_{A*B}, \beta, X) \subseteq \underline{U}(\nu_{A\circ B}, \beta, X)$.

Proof. This follows Lemma 7 and Proposition 5. $\square$

Proposition 7. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G with finite images and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If X is a non-empty subset of G, then

(i) $\overline{U}(\mu_{A\circ B}, \alpha, X) = \overline{U}(\mu_{A*B}, \alpha, X)$ and $\overline{U}(\nu_{A\circ B}, \beta, X) = \overline{U}(\nu_{A*B}, \beta, X)$.

(ii) $\underline{U}(\mu_{A\circ B}, \alpha, X) = \underline{U}(\mu_{A*B}, \alpha, X)$ and $\underline{U}(\nu_{A\circ B}, \beta, X) = \underline{U}(\nu_{A*B}, \beta, X)$.

Proof. This follows Lemma 8. $\square$

4 Approximations of IFS based on IFNSGs

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. By Lemma 2, $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$ is a congruence relation on G. Therefore, when $U = G$ and $\theta = U(A, \alpha, \beta)$, then we use $(G, A, \alpha, \beta) = ((G, \mu_A, \alpha), (G, \nu_A, \beta))$ instead of approximation space (U, θ).

Definition 9. *Let $U(A, \alpha, \beta) = U(\mu_A, \alpha) \cap U(\nu_A, \beta)$ is a congruence relation of an IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. Let $B = (\mu_B, \nu_B)$ be an IFS of G. Then we define the fuzzy sets $\underline{U}(\mu_A, \alpha, \mu_B)$, $\overline{U}(\mu_A, \alpha, \mu_B)$, $\underline{U}(\nu_A, \beta, \nu_B)$, $\overline{U}(\nu_A, \beta, \nu_B)$ as follows:*

$$\underline{U}(\mu_A, \alpha, \mu_B)(x) = \wedge_{a\in[x]_{(\mu_A,\alpha)}} \mu_B(a) \textit{ and } \overline{U}(\mu_A, \alpha, \mu_B)(x) = \vee_{a\in[x]_{(\mu_A,\alpha)}} \mu_B(a)$$
$$\underline{U}(\nu_A, \beta, \nu_B)(x) = \vee_{a\in[x]_{(\nu_A,\beta)}} \nu_B(a) \textit{ and } \overline{U}(\nu_A, \beta, \nu_B)(x) = \wedge_{a\in[x]_{(\nu_A,\beta)}} \nu_B(a).$$

The fuzzy sets $\underline{U}(\mu_A,\alpha,\mu_B)$ *and* $\overline{U}(\mu_A,\alpha,\mu_B)$ *are called, respectively, the* $U(\mu_A,\alpha)$*-lower and* $U(\mu_A,\alpha)$*-upper approximations of the fuzzy set* μ_B *with respect to* $U(\mu_A,\alpha)$*; and the fuzzy sets* $\underline{U}(\nu_A,\beta,\nu_B)$ *and* $\overline{U}(\nu_A,\beta,\nu_B)$ *are called, respectively, the* $U(\nu_A,\beta)$*-lower and* $U(\nu_A,\beta)$*-upper approximations of the fuzzy set* ν_B *with respect to* $U(\nu_A,\beta)$*; and the IFSs* $\underline{U}(A,\alpha,\beta,B) = (\underline{U}(\mu_A,\alpha,\mu_B),\underline{U}(\nu_A,\beta,\nu_B))$ *and* $\overline{U}(A,\alpha,\beta,B) = (\overline{U}(\mu_A,\alpha,\mu_B),\overline{U}(\nu_A,\beta,\nu_B))$ *are called, respectively, the* $U(A,\alpha,\beta)$*-lower and* $U(A,\alpha,\beta)$*-upper approximations of the IFS* $B=(\mu_B,\nu_B)$ *with respect to* $U(A,\alpha,\beta) = U(\mu_A,\alpha)\cap U(\nu_A,\beta)$*.*

$U(A,\alpha,\beta,B) = (\underline{U}(A,\alpha,\beta,B),\overline{U}(A,\alpha,\beta,B))$ *is called a rough IFS with respect to* $U(A,\alpha,\beta) = U(\mu_A,\alpha)\cap U(\nu_A,\beta)$ *if* $\underline{U}(A,\alpha,\beta,B) \neq \overline{U}(A,\alpha,\beta,B)$*.*

The following proposition is exactly obtained from Definition 9.

Proposition 8. For every approximation space $(G,A,\alpha,\beta) = ((G,\mu_A,\alpha),(G,\nu_A,\beta))$ and every IFSs $B=(\mu_B,\nu_B)$ and $C=(\mu_C,\nu_C)$ of G we have:

(1) $\underline{U}(\mu_A,\alpha,\mu_B)\subseteq\mu_B\subseteq\overline{U}(\mu_A,\alpha,\mu_B)$ and $\underline{U}(\nu_A,\beta,\nu_B)\supseteq\nu_B\supseteq\overline{U}(\nu_A,\beta,\nu_B)$.

(2) If $B\subseteq C$, then

$$\underline{U}(\mu_A,\alpha,\mu_B)\subseteq\underline{U}(\mu_A,\alpha,\mu_C) \text{ and } \underline{U}(\nu_A,\beta,\nu_B)\supseteq\underline{U}(\nu_A,\beta,\nu_C).$$

(3) If $B\subseteq C$, then

$$\overline{U}(\mu_A,\alpha,\mu_B)\subseteq\overline{U}(\mu_A,\alpha,\mu_C) \text{ and } \overline{U}(\nu_A,\beta,\nu_B)\supseteq\overline{U}(\nu_A,\beta,\nu_C).$$

(4) $\underline{U}(\mu_A,\alpha,(\underline{U}(\mu_A,\alpha,\mu_B)) = \underline{U}(\mu_A,\alpha,\mu_B)$ and $\underline{U}(\nu_A,\beta,(\underline{U}(\nu_A,\beta,\nu_B)) = \underline{U}(\nu_A,\beta,\nu_B)$.

(5) $\overline{U}(\mu_A,\alpha,(\overline{U}(\mu_A,\alpha,\mu_B)) = \overline{U}(\mu_A,\alpha,\mu_B)$ and $\overline{U}(\nu_A,\beta,(\overline{U}(\nu_A,\beta,\nu_B)) = \overline{U}(\nu_A,\beta,\nu_B)$.

(6) $\overline{U}(\mu_A,\alpha,((\underline{\mu}_A,\alpha,\mu_B)) = \underline{U}(\mu_A,\alpha,\mu_B)$ and $\overline{U}(\nu_A,\beta,(\underline{U}(\nu_A,\beta,\nu_B)) = \underline{U}(\nu_A,\beta,\nu_B)$.

(7) $\underline{U}(\mu_A,\alpha,(\overline{U}(\mu_A,\alpha,\mu_B)) = \overline{U}(\mu_A,\alpha,\mu_B)$ and $\underline{U}(\nu_A,\beta,(\overline{U}(\nu_A,\beta,\nu_B)) = \overline{U}(\nu_A,\beta,\nu_B)$.

(8) $\underline{U}(\mu_A,\alpha,\mu_B\wedge\mu_C) = \underline{U}(\mu_A,\alpha,\mu_B)\wedge\underline{U}(\mu_A,\alpha,\mu_C)$ and $\underline{U}(\nu_A,\beta,\nu_B\vee\nu_C) = \underline{U}(\nu_A,\beta,\nu_B)\vee\underline{U}(\nu_A,\beta,\nu_C)$.

(9) $\overline{U}(\mu_A,\alpha,\mu_B\wedge\mu_C) = \overline{U}(\mu_A,\alpha,\mu_B)\wedge\overline{U}(\mu_A,\alpha,\mu_C)$ and $\overline{U}(\nu_A,\beta,\nu_B\vee\nu_C) = \overline{U}(\nu_A,\beta,\nu_B)\vee\overline{U}(\nu_A,\beta,\nu_C)$.

(10) $\underline{U}(\mu_A,\alpha,\mu_B\vee\mu_C) = \underline{U}(\mu_A,\alpha,\mu_B)\vee\underline{U}(\mu_A,\alpha,\mu_C)$ and $\underline{U}(\nu_A,\beta,\nu_B\wedge\nu_C) = \underline{U}(\nu_A,\beta,\nu_B)\wedge\underline{U}(\nu_A,\beta,\nu_C)$.

(11) $\overline{U}(\mu_A,\alpha,\mu_B\vee\mu_C) = \overline{U}(\mu_A,\alpha,\mu_B)\vee\overline{U}(\mu_A,\alpha,\mu_C)$ and $\overline{U}(\nu_A,\beta,\nu_B\wedge\nu_C) = \overline{U}(\nu_A,\beta,\nu_B)\wedge\overline{U}(\nu_A,\beta,\nu_C)$.

(12) $\underline{U}(\mu_A,\alpha,\mu_B)(x) = \underline{U}(\mu_A,\alpha,\mu_B)(a)$ and $\overline{U}(\mu_A,\alpha,\mu_B)(x) = \overline{U}(\mu_A,\alpha,\mu_B)(a)$ for all $x\in G$ and $a\in[x]_{(\mu_A,\alpha)}$.

(13) $\underline{U}(\nu_A, \beta, \nu_B)(x) = \underline{U}(\nu_A, \beta, \nu_B)(a)$ and $\overline{U}(\nu_A, \beta, \nu_B)(x) = \overline{U}(\nu_A, \beta, \nu_B)(a)$ for all $x \in G$ and $a \in [x]_{(\nu_A, \beta)}$.

Proposition 9. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $B = (\mu_B, \nu_B)$ be any IFS of G and X a non-empty subset of G, then

(i) $\underline{U}(\mu_A, \alpha, \mu_B)(\overline{U}(\mu_A, \alpha, X)) = \underline{U}(\mu_A, \alpha, \mu_B)(X)$
and $\underline{U}(\nu_A, \beta, \nu_B)(\overline{U}(\nu_A, \beta, X)) \subseteq \underline{U}(\nu_A, \beta, \nu_B)(X)$.

(ii) $\overline{U}(\mu_A, \alpha, \mu_B)(\overline{U}(\mu_A, \alpha, X)) = \overline{U}(\mu_A, \alpha, \mu_B)(X)$
and $\overline{U}(\nu_A, \beta, \nu_B)(\overline{U}(\nu_A, \beta, X)) \subseteq \overline{U}(\nu_A, \beta, \nu_B)(X)$.

(iii) $\underline{U}(\mu_A, \alpha, \mu_B)(\underline{U}(\mu_A, \alpha, X)) \subseteq \underline{U}(\mu_A, \alpha, \mu_B)(X)$
and $\underline{U}(\nu_A, \beta, \nu_B)(\overline{U}(\nu_A, \beta, X)) = \underline{U}(\nu_A, \beta, \nu_B)(X)$.

(iv) $\overline{U}(\mu_A, \alpha, \mu_B)(\underline{U}(\mu_A, \alpha, X)) \subseteq \underline{U}(\mu_A, \alpha, \mu_B)(X)$
and $\overline{U}(\nu_A, \beta, \nu_B)(\underline{U}(\nu_A, \beta, X)) = \overline{U}(\nu_A, \beta, \nu_B)(X)$.

Where for every non-empty subset Y of G, $\underline{U}(\mu_A, \alpha, \mu_B)(Y) = \{\underline{U}(\mu_A, \alpha, \mu_B)(y) : y \in Y\}$ and $\underline{U}(\nu_A, \beta, \nu_B)(Y) = \{\underline{U}(\nu_A, \beta, \nu_B)(y) : y \in Y\}$; $\overline{U}(\mu_A, \alpha, \mu_B)(Y) = \{\overline{U}(\mu_A, \alpha, \mu_B)(y) : y \in Y\}$ and $\overline{U}(\nu_A, \beta, \nu_B)(Y) = \{\overline{U}(\nu_A, \beta, \nu_B)(y) : y \in Y\}$.

Proof. (i) Since $\overline{U}(\mu_A, \alpha, X) = \cup_{x \in X}[x]_{(\mu_A, \alpha)}$. We can conclude from the Proposition 8(12), that $\underline{U}(\mu_A, \alpha, \mu_B)(\underline{U}(\mu_A, \alpha, X)) = \cup_{x \in X}\underline{U}(\mu A, \alpha, \mu B)([x]_{(\mu_A, \alpha)}) = \underline{U}(\mu_A, \alpha, \mu_B)(X)$. Then by Proposition 1(1), it is straightforward.

The proof of the other part is similar to the proof of statement (i).

Proposition 10. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is an IFS of G, then

$\overline{U}(\mu_A \wedge \mu_B, \alpha, \mu_C) = \overline{U}(\mu_A, \alpha, \mu_C) \wedge \overline{U}(\mu_B, \alpha, \mu_C)$ and $\underline{U}(\nu_A \wedge \nu_B, \beta, \nu_C) = \underline{U}(\nu_A, \beta, \nu_C) \wedge \underline{U}(\nu_B, \beta, \nu_C)$.

Proof. For $x \in G$, we have

$$\begin{aligned}
&\overline{U}(\mu_A \wedge \mu_B, \alpha, \mu_C)(x) = \vee_{a \in [x]_{(\mu_A \wedge \mu_B, \alpha)}} \mu_C(a) \\
&= \vee_{a \in [x]_{(\mu_A, \alpha)}} \mu_C(a) \wedge \vee_{a \in [x]_{(\mu_B, \alpha)}} \mu_C(a) \\
&= \overline{U}(\mu_A, \alpha, \mu_C) \wedge \overline{U}(\mu_B, \alpha, \mu_C).
\end{aligned}$$

The rest is similar. □

Proposition 11. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is an IFS of G, then

$\underline{U}(\mu_A \wedge \mu_B, \alpha, \mu_C) = \underline{U}(\mu_A, \alpha, \mu_C) \wedge \underline{U}(\mu_B, \alpha, \mu_C)$ and $\overline{U}(\nu_A \wedge \nu_B, \beta, \nu_C) = \overline{U}(\nu_A, \beta, \nu_C) \wedge \overline{U}(\nu_B, \beta, \nu_C)$.

Proof. The proof is similar to the Proposition 10. □

Proposition 12. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. If $B = (\mu_B, \nu_B)$ be any IFNSG of G, then $\overline{U}(A, \alpha, \beta, B) = (\overline{U}(\mu_A, \alpha, \mu_B), \overline{U}(\nu_A, \beta, \nu_B))$ is an IFNSG of G.

Proof. For all $x, y \in G$, we have

$$\begin{aligned}\text{(i) } \overline{U}(\mu_A, \alpha, \mu_B)(xy) &= \vee_{z \in [xy]_{(\mu_A, \alpha)}} \mu_B(z) \\ &\geq \vee_{a \in [x]_{(\mu_A,\alpha)}, b \in [y]_{(\mu_A,\alpha)}} (\mu_B(a) \wedge \mu_B(b)) \\ &= \vee_{a \in [x]_{(\mu_A,\alpha)}} \mu_B(a) \wedge \vee_{b \in [y]_{(\mu_A,\alpha)}} \mu_B(b) \\ &= \overline{U}(\mu_A, \alpha, \mu_B)(x) \wedge \overline{U}(\mu_A, \alpha, \mu_B)(y).\end{aligned}$$

$$\text{(ii) } \overline{U}(\mu_A, \alpha, \mu_B)(x^{-1}) = \vee_{a \in [x^{-1}]_{(\mu_A,\alpha)}} \mu_B(a) = \vee_{a^{-1} \in [x]_{(\mu_A,\alpha)}} \mu_B(a^{-1}) = \overline{U}(\mu_A, \alpha, \mu_B)(x).$$

$$\begin{aligned}\text{(iii) } \overline{U}(\nu_A, \beta, \nu_B)(xy) &= \wedge_{z \in [xy]_{(\nu_A, \beta)}} \nu_B(z) \\ &\leq \wedge_{a \in [x]_{(\nu_A,\beta)}, b \in [y]_{(\nu_A,\beta)}} (\nu_B(a) \vee \nu_B(b)) \\ &= \wedge_{a \in [x]_{(\nu_A,\beta)}} \nu_B(a) \vee \wedge_{b \in [y]_{(\nu_A,\beta)}} \nu_B(b) \\ &= \overline{U}(\nu_A, \beta, \nu_B)(x) \vee \overline{U}(\nu_A, \beta, \nu_B)(y).\end{aligned}$$

$$\text{(iv) } \overline{U}(\nu_A, \beta, \nu_B)(x^{-1}) = \wedge_{a \in [x^{-1}]_{(\nu_A,\beta)}} \nu_B(a) = \wedge_{a^{-1} \in [x]_{(\nu_A,\beta)}} \nu_B(a^{-1}) = \overline{U}(\nu_A, \beta, \nu_B)(x).$$

Also, $\overline{U}(\mu_A, \alpha, \mu_B)(yx) = \vee_{z \in [yx]_{(\mu_A,\alpha)}} \mu_B(z) = \vee_{xzx^{-1} \in [xy]_{(\mu_A,\alpha)}} \mu_B(xzx^{-1}) = \overline{U}(\mu_A, \alpha, \nu_B)(xy)$ and $\overline{U}(\nu_A, \beta, \nu_B)(yx) = \wedge_{z \in [yx]_{(\nu_A,\beta)}} \nu_B(z) = \wedge_{xzx^{-1} \in [xy]_{(\nu_A,\beta)}} \nu_B(xzx^{-1}) = \overline{U}(\mu_A, \alpha, \mu_B)(xy)$. □

Proposition 13. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. If $B = (\mu_B, \nu_B)$ be any IFNSG of G, then $\underline{U}(A, \alpha, \beta, B) = (\underline{U}(\mu_A, \alpha, \mu_B), \underline{U}(\nu_A, \beta, \nu_B))$ is an IFNSG of G.

Proof. The proof is similar to the Proposition 12. □

Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha+\beta \leq 1$; and $(\underline{U}(A, \alpha, \beta, B), \overline{U}(A, \alpha, \beta, B))$ a rough IFS in the approximation space (G, A, α, β). If $\underline{U}(A,\alpha, \beta, B) = (\underline{U}(\mu_A, \alpha, \mu_B),\underline{U}(\nu_A, \beta, \nu_B))$ and $\overline{U}(A, \alpha, \beta, B) = (\overline{U}(\mu_A, \alpha, \mu_B), \overline{U}(\nu_A, \beta, \nu_B))$ are normal subgroup of G, then we call $(\underline{U}(A, \alpha, \beta, B), \overline{U}(A, \alpha, \beta, B))$ a rough IFNSG. Therefore, from Propositions 12 & 13, we have the following corollary.

Corollary 4. Let $A = (\mu_A, \nu_A)$ be an IFNSG of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. If $B = (\mu_B, \nu_B)$ be any IFNSG of G, then $(\underline{U}(A, \alpha, \beta, B), \overline{U}(A, \alpha, \beta, B))$ a rough IFNSG.

Proposition 14. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is IFS of G and $A \subseteq B$, then

(i) $\overline{U}(\mu_A, \alpha, \mu_C) \subseteq \overline{U}(\mu_B, \alpha, \mu_C)$ and $\overline{U}(\nu_A, \beta, \nu_C) \subseteq \overline{U}(\nu_B, \beta, \nu_C)$.

(ii) $\underline{U}(\mu_B, \alpha, \mu_C) \subseteq \underline{U}(\mu_A, \alpha, \mu_C)$ and $\underline{U}(\nu_B, \beta, \nu_C) \subseteq \underline{U}(\nu_A, \beta, \nu_C)$.

Proof. By Lemma 6, it is straightforward. $\square$

Proposition 15. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is a IFS of G and $U(A, \alpha, \beta) \subseteq U(B, \alpha, \beta)$ (i.e., $U(\mu_A, \alpha) \subseteq U(\mu_B, \alpha)$ and $U(\nu_A, \beta) \subseteq U(\nu_B, \beta)$) then,

(i) $\overline{U}(\mu_A, \alpha, \mu_C) \subseteq \overline{U}(\mu_B, \alpha, \mu_C)$ and $\overline{U}(\nu_A, \beta, \nu_C) \subseteq \overline{U}(\nu_B, \beta, \nu_C)$.

(ii) $\underline{U}(\mu_B, \alpha, \mu_C) \subseteq \underline{U}(\mu_A, \alpha, \mu_C)$ and $\underline{U}(\nu_B, \beta, \nu_C) \subseteq \underline{U}(\nu_A, \beta, \nu_C)$.

Proof. It is straightforward. $\square$

Proposition 16. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFNSGs of G and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is a IFS of G, then

(i) $\overline{U}(\mu_{A\circ B}, \alpha, \mu_C) \subseteq \overline{U}(\mu_{A*B}, \alpha, \mu_C)$ and $\overline{U}(\nu_{A\circ B}, \beta, \nu_C) \subseteq \overline{U}(\nu_{A*B}, \beta, \nu_C)$.

(ii) $\underline{U}(\mu_{A*B}, \alpha, \mu_C) \subseteq \underline{U}(\mu_{A\circ B}, \alpha, \mu_C)$ and $\underline{U}(\nu_{A*B}, \beta, \nu_C) \subseteq \underline{U}(\nu_{A\circ B}, \beta, \nu_C)$.

Proof. This follows Lemma 7 and Proposition 15. $\square$

Proposition 17. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two IFNSGs of G with finite images and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. If $C = (\mu_C, \nu_C)$ is IFS of G, then

(i) $\overline{U}(\mu_{A\circ B}, \alpha, \mu_C) = \overline{U}(\mu_{A*B}, \alpha, \mu_C)$ and $\overline{U}(\nu_{A\circ B}, \beta, \nu_C) = \overline{U}(\nu_{A*B}, \beta, \nu_C)$.

(ii) $\underline{U}(\mu_{A\circ B}, \alpha, \mu_C) = \underline{U}(\mu_{A*B}, \alpha, \mu_C)$ and $\underline{U}(\nu_{A\circ B}, \beta, \nu_C) = \underline{U}(\nu_{A*B}, \beta, \nu_C)$.

Proof. This follows Lemma 8 and Proposition 15. $\square$

5 Conclusions

This paper is devoted to the theoretical study of the rough set and intuitionistic fuzzy rough set within the context of intuitionistic fuzzy normal subgroup. In fact, the study concerns the relationship between three research topic, rough sets, groups and intuitionistic fuzzy sets, each of them have applications across many fields. We have presented a definition of the lower and upper approximations of a non empty crisp subsets and intuitionistic fuzzy sets of a group with respect to an intuitionistic fuzzy normal subgroup. Based on these results, we will study rough sets and intuitionistic fuzzy rough sets with respect to an intuitionistic fuzzy ideal in the next paper.

References

[1] Atanassov, K. T. (1986). Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87–96.

[2] Atanassov, K. T. (1994). New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61, 137–142.

[3] Atanassova, L. (2007). Fuzzy version of L. Zadeh's extension principle, *Notes on Intuitionistic Fuzzy Sets*, 13(3), 33–36.

[4] Banerjee, B., & Basnet Kr. D. (2003). Intuitionistic fuzzy subrings and ideals, *The Journal of fuzzy mathematics*, 11(1), 139–155.

[5] Biswas, R., Nanda, S. (1994). Rough groups and rough subgroups, *Bull. Polish Acad. Sci. Math.*, 42, 251–254.

[6] Davvaz, B. (2004). Roughness in rings, *Information Sciences*, 164, 147–163.

[7] Davvaz, B., Dudak, W. A., & Jun, Y. B. (2006). Intuitionistic fuzzy HV-submodules, *Information Sciences*, 176, 285–300.

[8] Davvaz, B., (2006). Roughness based on fuzzy ideals, *Information Sciences*, 176, 2417–2437.

[9] Davvaz, B., & Mahdavipour, M. (2006). Roughness in modules, *Information Sciences*, 176, 3658–3674.

[10] Dubois, D., & Prade, H. (1990). Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems*, 17(2 3), 191–209.

[11] Estaji, A. A., Khodaii, S., & Bahrami, S. (2011). On rough set and fuzzy sublattice, *Information Sciences*, 181, 3981–3994

[12] Hur, K., Kang, H. W., & Song, H. K. (2003). Intuitionistic fuzzy subgroups and subrings, *Honam Mathematical Journal*, 25, 19–41.

[13] Kuroki, N., & Wang, P. P. (1996). The lower and upper approximations in a fuzzy group, *Information Sciences*, 90, 203–220.

[14] Kuroki, N., & Mordeson, J. N. (1997). Structure of rough sets and rough groups, *The Journal of fuzzy mathematics*, 5(1), 183–191.

[15] Kuroki, N. (1997). Rough ideals in semigroups, *Information Sciences*, 100, 139–163.

[16] Pawlak, Z. (1982). Rough sets, *International Journal of Computer and Information Sciences*, 11, 341–356.

[17] Xiao, Q. M., & Zhang, Z. L. (2008). Rough prime ideals and rough fuzzy prime ideals in semigroups, *Information Sciences*, 178, 425–438.

[18] Zadeh, L. A. (1965). Fuzzy sets, *Information and Control*, 8 338–353.