Second Int. Conf. on IFS, Sofia, 3-4 Oct. 1998
NIFS 4 (1998) 2, 42-46

FUNDAMENTAL JUSTIFICATION OF INTUITIONISTIC FUZZY LOGIC AND OF INTERVAL-VALUED FUZZY METHODS

Misha Koshelev[1],
Vladik Kreinovich[2], Bhuvan Rachamreddy[2],
Haris Yasemis[2], and Krassimir T. Atanassov[3]

[1]3 Ames Street Box #57
Massachusetts Institute of Technology (MIT)
Cambridge, MA 02138, USA, mkosh@mit.edu

[2]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA, vladik@cs.utep.edu
[3]CLBME - Bulgarian Academy of Sciences
Sofia-1113, P.O.Box 12, Bulgaria,
krat@bgcict.acad.bg

Abstract. Traditional fuzzy logic uses a real number $d(S)$ from the interval $[0,1]$ to represent a person's degree of certainty in a statement S. There exist different methods of eliciting these degrees; most of these methods are based on the assumption that a person is able, for every two statements, to choose a statement with the larger degree of certainty. In real life, people are not always capable of a meaningful choice; as a result, instead of numerical values, we get *intervals.*

For example, in intuitionistic fuzzy logic, the degree of confidence is described by *two* numbers: $d^+(S)$ represents the degree of certainty in S, while $d^-(S)$ represents the degree of certainty in its negation $\neg S$. This can be expressed as an interval $\mathbf{d}(S) = [d^+(S), 1 - d^-(S)]$ of possible values of degree of certainty.

In this paper, we show that there is a fundamental reason for this inability, and thus, the use of interval-valued degrees of belief in intuitionistic fuzzy logic is justified.

Introduction. Traditional fuzzy logic uses a real number $d(S)$ from the interval $[0,1]$ to represent a person's degree of certainty in a statement S. There exist different methods of eliciting these degrees; most of these methods set a *scale* of statements $S(d)$ with different degrees of certainty $d \in [0,1]$, and then establish the degree of certainty by asking a person to compare his degree of certainty in a given statement S with his degree of certainty in different statements $S(d)$ from the given scale. If a person is able, for every two statements, to choose a statement with the larger degree of certainty, then we will be able to find, with an arbitrary accuracy, the value d for which S is equivalent to $S(d)$ ($S \sim S(d)$). This d is then returned as the desired degree of certainty $d(S)$ in a given statement S.

In real life, people are not always capable of a meaningful choice; as a result, instead of numerical values, we get *intervals.*

For example, in intuitionistic fuzzy logic [2], — from the interval analysis point of view, — the degree of confidence is described by *two* numbers: $d^+(S)$ represents the degree of certainty in S, while $d^-(S)$ represents the degree of certainty in its negation $\neg S$. In some cases, the only reason for uncertainty in S is the existence of arguments in favor of $\neg S$; in such cases, $d^+(S) = 1 - d^-(S)$, and the two numbers are, in effect, representable by a single degree of certainty — $d^-(S) = 1 - d^+(S)$ (?) —. In many other cases, however, there are other reasons for not believing in S than believing in $\neg S$, so we have $d^+(S) < 1 - d^-(S)$. In other words, instead of *single* value of degree of certainty, we have an *interval* $\mathbf{d}(S) = [d^+(S), 1 - d^-(S)]$ of possible values of degree of certainty.

The question is: Is this inability to compare degrees of certainty a drawback of *certain* experts, or is it a fundamental feature of *all* experts? To answer this question, let us consider a natural method of eliciting subjective probabilities, and show that within this elicitation method, interval values are inevitable.

An example of a scale. A natural example of — a — scale is a scale of so-called *subjective probabilities* (see, e.g., [4, 7, 8]): We first choose two alternatives: one good (denoted by B) and one clearly bad (denoted by B). Then, for each number d from the interval $[0, 1]$, we set up a lottery $L(d)$ in which G appears with probability d, and as $S(d)$, we choose a statement that G will be the result of a lottery $L(d)$.

For $d = 1$, the lottery $L(1)$ coincides with G; for $d = 0$, the lottery $L(0)$ leads to B with probability 1, and thus, coincides with B. The larger the probability of getting G, the better are the chances of winning the lottery. So, if $d > d'$, then $S(d) > S(d')$ (where $>$ means "more confident").

For this scale, as a degree of certainty of a statement S, we choose the probability d for which S is equivalent to a statement $S(d)$, i.e., for which lottery "if S then G else B" is equivalent, to an expert, to a lottery $L(d)$. This degree of certainty is also called a *subjective probability* of a statement S.

Binary search: a natural elicitation method. For any scale $S(d)$, we can determine the degree of certainty with respect to this scale fast if we use *binary search* (for a detailed description of this method and of its drawbacks, see, e.g., [3, 6]). Namely, we want to value $d = d(S)$ for which $S \sim S(d)$. Initially, the only thing we know about this degree d is that it is a number from the interval $[0, 1]$. On each step of the iterative process, we will try to get a narrower interval containing d.

Suppose that we know that the desired value $d = d(S)$ belongs to an interval $[\underline{d}, \overline{d}]$; in other words, we know that $S(\underline{d}) \leq S \leq S(\overline{d})$. Then, to narrow this interval further, we compute the midpoint $d_0 = (\underline{d} + \overline{d})/2$ of this interval, and ask a person (whose preferences we are describing) to compare S with the statement $S(d_0)$. Depending on the result of this comparison, we have three possibilities:

- If the person's confidence in S is larger than his confidence in $S(d_0)$, this means that $d = d(S) > d_0$, and therefore, that $p \in [d_0, \overline{d}]$.
- If the person's confidence in S is smaller than his confidence in $S(d_0)$, this means that $d = d(S) > d_0$, and therefore, that $p \in [\underline{d}, d_0]$.
- If the person has the same degree of confidence both in S and in $S(d_0)$, then we can conclude that $d = d(S) = d_0$. In this case, we have found $d(S)$, and no further search is necessary.

At every step, we decrease the width of the interval that contains d to a half. So, after s steps, we will include d into an interval of width 2^{-s}. In other words, we will get the value $d(S)$ with an accuracy of 2^{-s}.

If we fix this accuracy 2^{-s}, then we stop after s iterations.

In real life, the above-described schemes does not always work. The above method is based on the assumption that the person always comes up with a choice between two alternatives. In real life, a person is often reluctant to make a reasonable choice among the alternatives (see, e.g., [10]).

One possible way to overcome the person's reluctance is to *force* him or her to make a choice, and not take "I don't know" for an answer.

The main drawback of this solution to the "reluctance problem" is as follows: Suppose that we have three alternatives A_i, A_j, and A_k that are so similar to the person that he has trouble comparing them, although he is not sure that they are equivalent to him/her. Then, if the person is forced to make a choice between two of these three alternatives, then this choice will be made at random: e.g.,

- If we first ask the person to compare A_i and A_j, then he will pick, say A_i. We will interpret this choice as meaning that $A_i > A_j$.
- Then, we ask the same person to compare A_j and A_k. Again, this choice will be made at random. For example, it can end up with A_j being preferred to A_k, which we will interpret as $A_j > A_k$.
- Finally, after a certain number of other questions, during which the person has forgotten about his previous random-choice answers, we will ask this person to compare A_i and A_k. Again, this choice will be random, resulting, e.g., in $A_k > A_i$.

So, we have $A_i > A_j > A_k > A_i$: a contradiction.

There are some ways to deal with this inconsistent description (see, e.g., [1, 4, 5]), but maybe, a better way is not to force the person to make a decision and allow the person the possibility of the fourth ("I don't know") answer to a request to compare alternatives?

Enter intervals. This idea is implemented in the *interval* description of degrees of certainty (see, e.g., [10] and references therein).

In the binary search algorithm, at any given moment of time we are getting an interval $[\underline{d}, \overline{d}]$ that contains the actual value $d(S)$ of the degree of confidence. In the binary search algorithm, we assume that a person can always make a comparison, and therefore, that this interval can be made as narrow as we want. If at some step of this algorithm, comparison cannot be made, this means that we cannot narrow the resulting interval $[\underline{d}, \overline{d}]$ any further, and so, instead of a real number $d(S)$, we end up with an *interval* $[\underline{d}, \overline{d}]$ that characterizes the degree of certainty of a given person in the given statement A.

The problem. The problem that we will attack in this paper is as follows: *is the inability to compare alternatives fundamental, or is it caused by something that is not necessarily always present, e.g., by the limited computation abilities of human brains or by pure laziness of some decision makers whose preferences we want to describe?*

Solution. We will prove that it is impossible to have an algorithm that, given arbitrary two alternatives A and A', would always come up with one of the three possible answers:

- $A > A'$;
- $A < A'$;
- $A \sim A'$.

This impossibility is a *fundamental* reason why we are not always able to compare alternatives. Since this impossibility is fundamental, we thus justify the use of *interval-valued* degrees of certainty.

Comment. The description is informal, because we did not want to obscure the pretty simple idea by an unnecessary formalism. It is easy, however, to formalize this proof into a theorem.

Proof. We will prove this result by reduction to a contradiction. Let us assume that the algorithm whose existence we try to disprove actually exists. Let us denote this algorithm by u. Then, in particular, for each computer program $\mathcal{P}$ (e.g., written in Pascal, or in any other programming language), we will be able to apply this algorithm to the statement "the program $\mathcal{P}$ will eventually stop" (we will denote this statement by $S(\mathcal{P})$). In this case:

- if the program $\mathcal{P}$ halts, then the corresponding statement $S(\mathcal{P})$ is true, and thus, its subjective probability $d(S(\mathcal{P}))$ is equal to 1.
- if the program $\mathcal{P}$ does not halt, then the corresponding statement $S(\mathcal{P})$ is false, and thus, its subjective probability $d(S(\mathcal{P}))$ is equal to 0.

We have assumed that the algorithm u can compute, with an arbitrary accuracy, the degree of certainty $d(S)$ of each statement S. Thus, if we compute $d(S(\mathcal{P}))$ with an accuracy 1/4, we will get either a number $\leq 1/4$ (if $\mathcal{P}$ halts) or a number $\geq 3/4$ (if it does not). Thus, by comparing the approximate degree of certainty with 1/2, we will be able to check whether a given program $\mathcal{P}$ halts. It is, however, known that no such algorithm is possible (see, e.g., [9]).

Therefore our assumption that there exists an algorithm that can compute the degree of confidence in each statement is false.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209 and grant No. NAG 9-757, by NSF grants No. EEC-9322370 and DUE-9750858, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0564.

References

[1] K. Atanassov, "Intuitionistic fuzzy sets and expert estimations. II", *BUSEFAL*, Vol. 59, 1994, 64-69.

[2] K. Atanassov and G. Gargov, "Elements of intuitionistic fuzzy logic. I", *Fuzzy sets and Systems*, Vol. 95, 1998, No. 1, 39-52.

[3] K. Atkinson, *Elementary Numerical Analysis* (2nd ed), John Wiley & Sons Inc., New York, 1992.

[4] P. C. Fishburn, *Utility Theory for Decision Making*, John Wiley & Sons Inc., New York, 1969.

[5] O. M. Kosheleva and V. Kreinovich, "Algorithmic problems of nontransitive (SSB) utilities," *Mathematical Social Sciences*, 1991, Vol. 21, pp. 95–100.

[6] V. Kreinovich, A review of "Uncertain reasoning" by G. Shafer and J. Pearl (eds.), *SIGART Bulletin*, 1992, Vol. 3, No. 4, pp. 23–27.

[7] D. R. Luce and H. Raiffa, *Games and Decisions, Introduction and critical survey*, John Wiley & Sons, Inc., New York, 1957.

[8] R. B. Myerson, *Game theory. Analysis of conflict*, Harvard University Press, Cambridge, MA, 1991.

[9] C. H. Papadimitriou, *Computational Complexity*, Addison Wesley, San Diego, 1994.

[10] P. Suppes, D. M. Krantz, R. D. Luce, and A. Tversky, *Foundations of measurement. Vol. II. Geometrical, threshold, and probabilistic representations*, Academic Press, San Diego, CA, 1989.