

# Remark on Dworoniczak's intuitionistic fuzzy implications. Part 3

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**Abstract:** The paper is a continuation of previous author's research. In it, on the basis of third implication, introduced by Piotr Dworoniczak, four new intuitionistic fuzzy implications are defined. Some of their basic properties are studied.

**Keywords:** Intuitionistic fuzzy implication, Intuitionistic fuzzy set.

**AMS Classification:** 03E72.

## 1 Introduction

In [8–10], Piotr Dworoniczak introduced three intuitionistic fuzzy implications, that generalized the intuitionistic fuzzy implications, defined by the author in [3–5].

Here, continuing the idea and research from [6, 7], we introduce four new intuitionistic fuzzy implications, modifying Dworoniczak's intuitionistic fuzzy implication

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta} \langle c, d \rangle = \left\langle \frac{b + c + \alpha - 1}{\alpha + \beta}, \frac{a + d + \beta - 1}{\alpha + \beta} \right\rangle,$$

where  $\alpha \geq 1, \beta \in [1, \alpha]$ .

## 2 Main results

Let everywhere below variables  $x$  and  $y$  have truth values  $\langle a, b \rangle$  and  $\langle c, d \rangle$ , where  $a, b, c, d \in [0, 1], a + b \leq 1, c + d \leq 1$ .

In [1, 2] the following operations and operators are defined:

$$\neg \langle a, b \rangle \equiv \neg_1 \langle a, b \rangle = \langle b, a \rangle,$$

$$\langle a, b \rangle @ \langle c, d \rangle = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle,$$

$$\square \langle a, b \rangle = \langle a, 1-a \rangle,$$

$$\diamond \langle a, b \rangle = \langle 1-b, b \rangle,$$

$$\boxplus_{\alpha, \beta, \gamma} \langle a, b \rangle = \langle \alpha a, \beta b + \gamma \rangle,$$

$$\boxtimes_{\alpha, \beta, \gamma} \langle a, b \rangle = \langle \alpha a + \gamma, \beta b \rangle,$$

where  $\alpha, \beta, \gamma \in [0, 1]$  and  $\max(\alpha, \beta) + \gamma \leq 1$ ,

$$\blacksquare_{\alpha, \beta, \gamma, \delta} \langle a, b \rangle = \langle \alpha a + \gamma, \beta b + \delta \rangle,$$

where  $\alpha, \beta, \gamma, \delta \in [0, 1]$  and  $\max(\alpha, \beta) + \gamma + \delta \leq 1$ .

We use the following formulas:

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^1 \langle c, d \rangle = \square \langle a, b \rangle \rightarrow_{152, \alpha, \beta} \square \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^2 \langle c, d \rangle = \square \langle a, b \rangle \rightarrow_{152, \alpha, \beta} \diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^3 \langle c, d \rangle = \diamond \langle a, b \rangle \rightarrow_{152, \alpha, \beta} \diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^4 \langle c, d \rangle = \diamond \langle a, b \rangle \rightarrow_{152, \alpha, \beta} \square \langle c, d \rangle,$$

where  $\gamma \geq 1$ .

So, we obtain the explicit forms of the new four implications as follows:

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^1 \langle c, d \rangle = \left\langle \frac{-a+c+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \right\rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^2 \langle c, d \rangle = \left\langle \frac{1-a-d+\alpha}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta} \right\rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^3 \langle c, d \rangle = \left\langle \frac{b-d+\alpha}{\alpha+\beta}, \frac{-b+d+\beta}{\alpha+\beta} \right\rangle,$$

$$\langle a, b \rangle \rightarrow_{152, \alpha, \beta}^4 \langle c, d \rangle = \left\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{1-b-c+\beta}{\alpha+\beta} \right\rangle.$$

First, we check that for every  $i = 1, 2, 3, 4$  and for every  $\alpha, \beta$ , so that  $\alpha \geq 1, \beta \in [1, \alpha]$ :

$$\langle 0, 1 \rangle \rightarrow_{152, \alpha, \beta}^i \langle 0, 1 \rangle = \left\langle \frac{\alpha+1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta} \right\rangle,$$

$$\langle 0, 1 \rangle \rightarrow_{152, \alpha, \beta}^i \langle 1, 0 \rangle = \left\langle \frac{\alpha+1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta} \right\rangle,$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^i \langle 0, 1 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle.$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^i \langle 1, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle.$$

Second, we check that for every  $i = 1, 2, 3, 4$  and for every  $\alpha, \beta$ , so that  $\alpha \geq 1, \beta \in [1, \alpha]$ :

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^1 \langle 0, 1 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^2 \langle 0, 1 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^3 \langle 0, 1 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^4 \langle 0, 1 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^1 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^2 \langle 0, 0 \rangle = \left\langle \frac{\alpha + 1}{\alpha + \beta}, \frac{\beta - 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^3 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^4 \langle 0, 0 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^1 \langle 1, 0 \rangle = \left\langle \frac{\alpha + 1}{\alpha + \beta}, \frac{\beta - 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^2 \langle 1, 0 \rangle = \left\langle \frac{\alpha + 1}{\alpha + \beta}, \frac{\beta - 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^3 \langle 1, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 0 \rangle \rightarrow_{152,\alpha,\beta}^4 \langle 1, 0 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 1 \rangle \rightarrow_{152,\alpha,\beta}^1 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 1 \rangle \rightarrow_{152,\alpha,\beta}^2 \langle 0, 0 \rangle = \left\langle \frac{\alpha + 1}{\alpha + \beta}, \frac{\beta - 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 1 \rangle \rightarrow_{152,\alpha,\beta}^3 \langle 0, 0 \rangle = \left\langle \frac{\alpha + 1}{\alpha + \beta}, \frac{\beta - 1}{\alpha + \beta} \right\rangle,$$

$$\langle 0, 1 \rangle \rightarrow_{152,\alpha,\beta}^4 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^1 \langle 0, 0 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle,$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^2 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^3 \langle 0, 0 \rangle = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle,$$

$$\langle 1, 0 \rangle \rightarrow_{152,\alpha,\beta}^4 \langle 0, 0 \rangle = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{\beta + 1}{\alpha + \beta} \right\rangle.$$

Using the definition from [1, 2]

$$\langle a, b \rangle \geq \langle c, d \rangle \text{ if and only if } a \geq c \text{ and } b \leq d,$$

we can prove the validity of the following theorem.

**Theorem 1.** For every  $a, b, c, d \in [0, 1]$ , so that  $a + b \leq 1$  and  $c + d \leq 1$  and for every  $\alpha \geq 1, \beta \in [1, \alpha]$ :

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^2 \langle c, d \rangle \geq \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^1 \langle c, d \rangle, \quad (1)$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^3 \langle c, d \rangle \geq \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^4 \langle c, d \rangle, \quad (2)$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^2 \langle c, d \rangle \geq \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^3 \langle c, d \rangle, \quad (3)$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^1 \langle c, d \rangle \geq \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^4 \langle c, d \rangle. \quad (4)$$

*Proof:* For example, let us check the validity of the fourth inequality. First, we see, that

$$1 - a + c + \alpha - (b + c + \alpha - 1) = 1 - a - b \geq 0,$$

$$1 - b - c + \beta - (a - c + \beta) = 1 - a - b \geq 0.$$

Therefore,

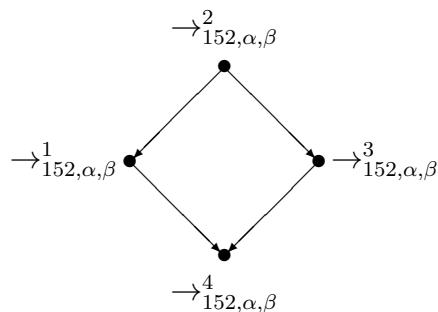
$$\left\langle \frac{1 - a + c + \alpha}{\alpha + \beta}, \frac{a - c + \beta}{\alpha + \beta} \right\rangle \geq \left\langle \frac{b + c + \alpha - 1}{\alpha + \beta}, \frac{1 - b - c + \beta}{\alpha + \beta} \right\rangle,$$

i.e.,

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^1 \langle c, d \rangle \geq \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^4 \langle c, d \rangle.$$

Hence, (4) is valid. Inequalities (1) – (3) are proved by analogical manner.

Now, we can construct the following diagram.



Now, we show that the new implications can be represented by a part of the above operators.

**Theorem 2.** For every  $a, b, c, d \in [0, 1]$ , so that  $a + b \leq 1$  and  $c + d \leq 1$  and for every  $\alpha \geq 1, \beta \in [1, \alpha]$ :

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^1 \langle c, d \rangle = \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \neg \langle a, b \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^2 \langle c, d \rangle = \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \neg \langle a, b \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^3 \langle c, d \rangle = \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \neg \langle a, b \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \rightarrow_{152,\alpha,\beta}^4 \langle c, d \rangle = \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \neg \langle a, b \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \langle c, d \rangle.$$

*Proof:* Let  $a, b, c, d \in [0, 1]$ , so that  $a + b \leq 1$  and  $c + d \leq 1$  and let  $\alpha \geq 1, \beta \in [1, \alpha]$ . Then

$$\begin{aligned} & \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \neg \langle a, b \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \langle c, d \rangle \\ &= \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \diamond \langle b, a \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \square \langle c, d \rangle \\ &= \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \langle 1 - a, a \rangle @ \bullet_{\frac{2}{\alpha+\beta}, \frac{2}{\alpha+\beta}, \frac{\alpha-1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta}} \langle c, 1 - c \rangle \\ &= \left\langle \frac{2(1-a)}{\alpha+\beta} + \frac{\alpha-1}{\alpha+\beta}, \frac{2a}{\alpha+\beta} + \frac{\beta-1}{\alpha+\beta} \right\rangle @ \left\langle \frac{2c}{\alpha+\beta} + \frac{\alpha-1}{\alpha+\beta}, \frac{2(1-c)}{\alpha+\beta} + \frac{\beta-1}{\alpha+\beta} \right\rangle \\ &= \left\langle \frac{1}{2} \left( \frac{2(1-a)}{\alpha+\beta} + \frac{\alpha-1}{\alpha+\beta} + \frac{2c}{\alpha+\beta} + \frac{\alpha-1}{\alpha+\beta} \right), \frac{1}{2} \left( \frac{2a}{\alpha+\beta} + \frac{\beta-1}{\alpha+\beta} + \frac{2(1-c)}{\alpha+\beta} + \frac{\beta-1}{\alpha+\beta} \right) \right\rangle \\ &= \left\langle \frac{1}{2} \cdot \frac{2 - 2a + \alpha - 1 + 2c + \alpha - 1}{\alpha+\beta}, \frac{1}{2} \cdot \frac{2a + \beta - 1 + 2 - 2c + \beta - 1}{\alpha+\beta} \right\rangle \\ &= \left\langle \frac{-a + c + \alpha}{\alpha+\beta}, \frac{a - c + \beta}{\alpha+\beta} \right\rangle \\ &= \langle a, b \rangle \rightarrow_{152,\alpha,\beta}^1 \langle c, d \rangle. \end{aligned}$$

The proofs of the rest assertions are similar to the first one.  $\square$

### 3 Conclusion

In the next parts of the research, we will study from one side other properties of the new implications and from another – the modifications of the third Dworczyk's implication and their properties.

We finish with the following two open problems.

**Open Problem 1:** Can all intuitionistic fuzzy implications be represented in a similar way by operator  $\bullet$ ?

**Open Problem 2:** Can the four new intuitionistic fuzzy implications be represented by similar way by operator  $\boxplus_{\alpha,\beta,\gamma}$  and  $\boxtimes_{\alpha,\beta,\gamma}$ , as it is done in [6] for the introduced there four intuitionistic fuzzy implications?

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