# FUZZY-RATIONAL EXPLANATION OF THE ELLSBERG PARADOX 

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Abstract: The paper analyses the Ellsberg paradox from the point of view of fuzzy rational decision makers, who can only partially measure uncertainty in terms of interval probabilities. Alternatives are modeled as fuzzy-rational lotteries, and are brought down to classical risky lotteries using intuitionistic operators according to a preliminarily chosen decision criterion under strict uncertainty. The Hurwicz $\alpha_{\alpha}$ expected utility criterion serves to prove that declared preferences in the Ellsberg paradox are consistent and reasonable, if the fuzzyrational decision maker is a moderate or extreme pessimist.

Keywords: Ellsberg paradox, ambiguity, fuzzy rationality, partially quantified uncertainty, interval probabilities, Hurwicz $\alpha_{\alpha}$ expected utility

## 1. Introduction

When making choices under uncertainty, each alternative may be modeled as a lotteries, which is a full group of disjoint events (called states) and the respective consequences (called prizes) from these events. Ranking alternatives according to preference is called task under strict uncertainty, if the only thing the decision maker (DM) can do is to differentiate states to possible and impossible. Several criteria are proposed to rank alternatives in that setup, namely Wald [Wald, 1950], Hurwicz ${ }_{\alpha}$ [Hurwicz, 1951], Savage [Savage, 1951] and Laplace criteria [Rapoport, 1989]. However, none of these leads to rational decisions. Ranking alternatives according to preference is called a task under risk if the DM is in position to measure the relative likelihood of all states in the alternatives in terms of probabilities. Then axioms and theorems of utility theory [von Neumann, Morgenstern, 1947] prove the existence of the utility function $u($.$) over prizes, whose values increase with the increase of preferences$ of the DM over prizes. The expected value of utility serves as a rational criterion to rank uncertain alternatives. Utilities and probabilities are subjective and typical for each DM [Bernstein, 1996].

Works document many paradoxes, i.e. systematic deviations of DM's preference from normative rules of utility theory. One of these is the Ellsberg paradox [Ellsberg, 1961], where in two consecutive tasks of choice between alternatives, most people demonstrate mutually contradicting, preferences in terms of utility theory, at arbitrary probabilities and utilities. There are two ways to explain the observer paradox.

The common opinion is that people are irrational in their behavior [Kahneman, Tversky, 1973], and obey heuristic and non-heuristic biases and fallacies [Tversky, Kahneman, 1974; Clemen, 1996]. For example, ambiguity [Fox, Tversky, 1995] (a tendency to bet on known
probabilities than to bet on gambles with "unknown" probabilities) is a non-heuristic bias that causes the Ellsberg paradox. Another cause of this paradox might be irrationality of subjective probability statements [Kahneman, Tversky, 1979].

There have been attempts to explain paradoxes by imperfections of utility theory, which is why it is replaced with other generalized new normative theoretical models. For example, the choice of individuals that complies with the Ellsberg paradox is reasonable and expected by the info-gap theory [Ben-Haim, 2006], by the anticipated utility approach [Segal, 1987], by Machina's functional [Machina, 1982], by Chew's weighted function [Chew, 1983] and by generalized expected utility approach [Quiggin, 1993]. Following these ideas, this paper analyzes the Ellsberg paradox from the point of view of a fuzzy rational DM, who chooses according to Hurwicz ${ }_{\alpha}$ expected utility. This criterion generalizes utility theory in cases when uncertainty is partially quantified in terms of interval probabilities.

Utility and probability elicitation procedures are relatively similar, and in both cases the DM must compare prizes and/or lotteries of certain structure according to preference. Elicitation techniques are based on several sets of rationality assumptions [Savage, 1954; De Groot, 1970], among which transitivity of binary relations of preference. These assumptions hold for the ideal DM, who identifies unique point estimates of utilities and probabilities due to her/his infinite discriminating abilities. The real DM partially disobeys transitivity of indifference and mutual transitivity of strict preference and indifference. That is why in [Nikolova, et al., 2005] such individuals are called fuzzy-rational DMs, and they identify interval estimates of utilities and probabilities. Probability intervals in each alternative are transformed into point estimates using intuitionistic operators, according to an irrational criterion under strict uncertainty. A task under risk is then defined and alternatives are ranked according to expected utility [French, Insua, 2000]. It is proven that if a pessimistic DM uses the Hurwicz $_{\alpha}$ criterion for the transformation of probabilities, then she/he would act according to Ellsberg paradox and shall exhibit ambiguity.

In what follows, section two comments the setup of the Ellsberg paradox and the associated ambiguity. Section three formulates problems under strict uncertainty and under risk, and discusses criteria to rank alternatives in those two cases. Fuzzy rationality is discussed in section four, as well as the resulting interval estimates of probabilities. Fuzzy rational lotteries are constructed on that basis and three criteria are proposed to rank those. A formalization of the Ellsberg paradox is given in section five, and the Hurwicz ${ }_{\alpha}$ expected utility criterion is used to prove that observed preferences of people are rational and consistent.

## 2. Ellsberg paradox

The formulation of the Ellsberg paradox is given in [Ellsberg, 1961]. An urn contains 90 balls, of which 30 are red, and the other 60 are either blue of yellow in unknown proportion. The DM has to choose between gambles $A$ and $B$, where $A$ is "to receive $\$ 1000$ if a red ball is drawn from the urn and nothing otherwise", and $B$ is "to receive $\$ 1000$ if a blue ball is drawn and nothing otherwise". A choice between gambles $C$ and $D$ is also proposed, where $C$ is "to receive $\$ 1000$ if a red or yellow ball is drawn and nothing otherwise", and $D$ is "to receive $\$ 1000$ if a blue or yellow ball is drawn and nothing otherwise". The works [Slovic, Tversky, 1974; MacCrimmon, Larsson, 1979] empirically prove the Ellsberg's hypothesis that most people would choose $A$ to $B$ (since $A$ gives a $1 / 3$ chance to win, whereas
at $B$ chances are vague and vary from 0 to $2 / 3$ ), and at the same time would choose $D$ to $C$ (since at $D$ the chance to win is exactly $2 / 3$, whereas at $C$ chances to win are vague and vary from $1 / 3$ to 1 ).

Such pair of choices contradicts the independence axiom of subjective probability [French, 1993]. Let $G, S$ and $T$ are three random events, such that $G \cap T=\mathscr{O}$ and $S \cap T=\mathscr{O}$. Then the axiom states that if the DM holds $G \succsim_{l} S$ (where $\succeq_{l}$ denotes the binary relation "at leas as likely as"), then she/he should also hold $G \cup T \succsim_{l} S \cup T$. Statements of similar structure may be formulated for the binary relations "equally likely" $\left(\sim_{l}\right)$ and "more likely than" $\left(\succ_{l}\right)$.

This axiom does not hold in the Ellsberg paradox, since preference of $A$ over $B$ means that the probability for a red ball is higher than the probability for a blue ball according to the DM. Drawing a yellow ball guarantees that neither red nor blue ball can be drawn. Then the probability for a red or yellow ball should still be higher than that of a blue and yellow ball. However, the preference of $D$ over $C$ states the opposite.

The tendency in people to bet on gambles with known probabilities than on gambles with "unknown" probabilities is called ambiguity [Ellsberg, 1961]. This effect is empirically studied in [Fox, Tversky, 1995]. Evidence showed that people prefer to bet on "unknown" statements in cases when they feel competent, and vise versa - prefer known probabilities when they do not feel competent on the matter. It is also proven that avoiding uncertainty "... is driven by the feeling of incompetence ... [and] will be present when subjects evaluate clear and vague prospects jointly, but it will greatly diminish or disappear when they evaluate each prospect in isolation". For example, people knowledgeable in politics and ignorant about sports would prefer betting on political events than betting on games of chance at the same odds, but would prefer betting on games of chance to betting on sport games under the same conditions.

## 3. Choosing alternatives under strict uncertainty and under risk

A formalization of tasks under strict uncertainty and under risk is proposed in [Tenekedjiev, 2006]. A DM should compare, according to preference, a set of $n$ uncertain alternatives that given $r$ different prizes (consequences) $x_{j}$, numbered so that $x_{1}$ is most preferred, and $x_{r}$ is the least preferred prize. Let's define the states $\theta_{i, 1}, \theta_{i, 2}, \ldots, \theta_{i, r}$, that are a set of hypotheses. If $\theta_{i, j}$ occurs under the $i$-th alternative, then the DM would receive $x_{j}$. Then the ordinary lottery may be represented as in (1) with conditions in (2):

$$
\begin{align*}
& l_{i}=<\theta_{i, 1}, x_{i} ; \theta_{i, 2}, x_{2} ; \ldots ; \theta_{i, r}, x_{r}>,  \tag{1}\\
& x_{1} \succsim x_{2} \succsim \ldots \succsim x_{r} \\
& \theta_{i, j} \succsim \varnothing  \tag{2}\\
& \theta_{i, 1} \cup \theta_{i, 2} \cup \ldots \cup \theta_{i, r}=\Theta, \\
& \theta_{i, j} \cap \theta_{i, k}=\varnothing, \text { for } j \neq k
\end{align*}
$$

Here, $\varnothing$ is the impossible event, $\Theta$ is the certain event, and $\succsim$ is the binary relation ,at least as preferred as" over prizes and lotteries. The DM has constructed a utility function $u($. over the prizes $x_{j}$, such that

$$
\begin{equation*}
u\left(x_{j}\right) \geq u\left(x_{k}\right) \Leftrightarrow x_{j} \succsim x_{k}, \tag{3}
\end{equation*}
$$

and with the conditions that $u\left(x_{1}\right)=1, u\left(x_{r}\right)=0$ [Keeney, Raiffa, 1993]. Under strict uncertainty, the DM only knows which states are possible and which are not. The discrete Boolean function $b($.$) over the states \theta_{i, j}$ may be defined:

$$
\begin{align*}
& b\left(\theta_{i, j}\right)= \begin{cases}'^{\prime} t^{\prime}, & \text { for } \theta_{i, j} \succ_{l} \varnothing \\
f^{\prime}, & \text { for } \theta_{i, j} \sim \\
\sim\end{cases}  \tag{4}\\
& b\left(\theta_{i, 1}\right) \vee b\left(\theta_{i, 2}\right) \vee \ldots \vee b\left(\theta_{i, r}\right)=^{\prime} t^{\prime} . \tag{5}
\end{align*}
$$

where $\vee$ is the Boolean operator "and". Then formulae (2), (5) and (6) describe lotteries under strict uncertainty:

$$
\begin{equation*}
l_{i}^{s u}=\ll \theta_{i, 1}, b\left(\theta_{i, 1}\right)>, x_{i} ;<\theta_{i, 2}, b\left(\theta_{i, 2}\right)>, x_{2} ; \ldots ;<\theta_{i, r}, b\left(\theta_{i, r}\right)>, x_{1}>. \tag{6}
\end{equation*}
$$

Lotteries (6) may be ranked using several criteria, among which Wald and Hurwicz ${ }_{\alpha}$ criteria.

Wald's maximin return criterion [Wald, 1950] introduces the security level $s_{i}$, which coincides with the worst consequence of the $i$-th alternative:

$$
\begin{equation*}
s_{i}=\min _{\substack{j=1 \\ b\left(\theta_{i, j}\right)==^{\prime} t^{\prime}}}^{r}\left\{u\left(x_{j}\right)\right\} . \tag{7}
\end{equation*}
$$

The recommended alternative is $l_{k}$, where the security level is maximal. That is why Wald's criterion is suitable for extreme pessimists.

If the DM is an extreme optimist, she/he can use the maximax criterion. The optimism level $o_{i}$ is introduced, which is the best possible outcome from the $i$-th alternative

$$
\begin{equation*}
o_{i}=\max _{\substack{j=1 \\ b\left(\theta_{i, j}\right)=\prime^{\prime} t^{\prime}}}^{r}\left\{u\left(x_{j}\right)\right\} . \tag{8}
\end{equation*}
$$

The recommended alternative is $l_{k}$, for which the optimism level is maximal. Since people are rarely extreme pessimists or optimists, the Hurwicz ${ }_{\alpha}$ criterion weighs the security level and the optimism level by the pessimism-optimism index $\alpha \in[0 ; 1]$ [Hurwicz, 1951]

$$
\begin{equation*}
h_{i}^{\alpha}=\alpha s_{i}+(1-\alpha) o_{i} . \tag{9}
\end{equation*}
$$

The recommended alternative is $l_{k}$, for which $h_{i}^{\alpha}$ is maximal. The value of $\alpha$ is a measure of pessimism of the DM. It is subjectively elicited and holds for all decision problems of that DM. A set of rationality conditions for each decision criterion is defined in
[French, 1993]. It is proven that none of the decision criteria under strict uncertainty is rational.

Under risk, the ideal DM defines unique point estimates of probabilities of $\theta_{i, j}$ :

$$
\begin{align*}
& P\left(\theta_{i, j}\right) \geq 0,  \tag{10}\\
& P\left(\theta_{i, 1}\right)+P\left(\theta_{i, 2}\right)+\ldots+P\left(\theta_{i, r}\right)=1 . \tag{11}
\end{align*}
$$

Then the lottery (1) may be brought down to (12), with conditions in (2), (10) and (11) and is called a classical risky lottery:

$$
\begin{equation*}
l_{i}^{c r}=\left\langle P\left(\theta_{i, 1}\right), x_{i} ; P\left(\theta_{i, 2}\right), x_{2} ; \ldots ; P\left(\theta_{i, r}\right), x_{r}>.\right. \tag{12}
\end{equation*}
$$

If certain rationality conditions hold [French, Insua, 2000], then these lotteries may be ranked in descending order of expected utility:

$$
\begin{equation*}
E_{i}(u / p)=\sum_{j=1}^{r} P\left(\theta_{i, j}\right) u_{j} . \tag{13}
\end{equation*}
$$

The recommended alternative is $l_{k}$, where $E_{i}(u / p)$ is maximal.
Rationality conditions may be defined together for preference and expectations [Villigas, 1964; Tenekedjiev, 2004; De Groot, 1970] or in separate axiomatic sets [Ramsay, 1931; Savage, 1954; Pratt, et al., 1995]. Transitivity of binary relations of preference is present in both cases.

## 4. Choosing between alternatives with partially quantified uncertainty

When eliciting the subjective probability of a random event $B$, the DM compares two gambles. The first gamble, $l_{1}(B)$, gives a huge prize if $B$ occurs and nothing otherwise. The other gamble, $l_{2}(m, n)$, is based on an urn of $n$ balls, of which $m$ are white, and the rest are black. The DM receives the same huge prize if she/he draws a white ball, and nothing otherwise. The preferential equation $l_{1}(B) \sim l_{2}(m, n)$ is solved according to $m$ at given $B$ and fixed $n$, using bisection [Press, et al., 1992]. When indifference is declared, $P(B)=m(B) / n$. This holds for the ideal DM. For the fuzzy-rational DM there exist two integers $m_{1}$ and $m_{2}$ ( $m_{2}>m_{1}$ ), such that $l_{1}(B) \sim l_{2}\left(m_{1}, n\right), l_{1}(B) \sim l_{2}\left(m_{2}, n\right), l_{2}\left(m_{2}, n\right) \succ l_{2}\left(m_{1}, n\right)$. That is why it is necessary to find the greatest $m=m_{\text {down }}$, where $l_{1}(B) \succ l_{2}\left(m_{\text {down }}, n\right)$, and the smallest $m=m_{u p}$, where $\quad l_{2}\left(m_{u p}, n\right) \succ l_{1}(B)$. Then the root $\quad m(B) \in\left(m_{\text {down }} ; m_{u p}\right) \quad$ and $\quad m_{\text {down }} / n=P_{d}(B)<$ $<P(B)<P_{u}(B)=m_{u p} / n$. Finally, $P(B) \in\left[P_{d}(B) ; P_{u}(B)\right]$, in order to accommodate the case when probabilities are known. The uncertainty interval of $P(B)$ is elicited using triple bisection [Tenekedjiev, et al., 2004].

If the DM can elicit the uncertainty interval of the probabilities of $\theta_{i, j}$, defined in (14) with conditions in (15), a problem with partially quantified uncertainty exists [Tenekedjiev, 2006]:

$$
\begin{equation*}
P\left(\theta_{i, j}\right) \in\left[P_{d}\left(\theta_{i, j}\right) ; P_{u}\left(\theta_{i, j}\right)\right], \text { for } i=1,2, \ldots, n \text { and } j=1,2, \ldots, r, \tag{14}
\end{equation*}
$$

$$
\left\lvert\, \begin{align*}
& P_{d}\left(\theta_{i, j}\right) \geq 0, \text { for } j=1,2, \ldots, r,  \tag{15}\\
& P_{d}\left(\theta_{i, j}\right) \leq P_{u}\left(\theta_{i, j}\right), \text { for } j=1,2, \ldots, r, \\
& P_{u}\left(\theta_{i, j}\right) \leq 1, \text { for } j=1,2, \ldots, r, \\
& P_{d}\left(\theta_{i, 1}\right)+P_{d}\left(\theta_{i, 2}\right)+\ldots+P_{d}\left(\theta_{i, r}\right) \leq 1, \\
& P_{u}\left(\theta_{i, 1}\right)+P_{u}\left(\theta_{i, 2}\right)+\ldots+P_{u}\left(\theta_{i, r}\right) \geq 1 .
\end{align*}\right.
$$

The lottery (1) with probabilities in (14) is called a fuzzy-rational lottery:

$$
\begin{align*}
l_{i}^{f r} & =\ll \theta_{i, 1}, P_{d}\left(\theta_{i, 1}\right), 1-P_{u}\left(\theta_{i, 1}\right)>, x_{i} ;<\theta_{i, 2}, P_{d}\left(\theta_{i, 2}\right), 1-P_{u}\left(\theta_{i, 2}\right)>, x_{2} ; \ldots ; \\
& <\theta_{i, r}, P_{d}\left(\theta_{i, r}\right), 1-P_{u}\left(\theta_{i, r}\right)>, x_{r}>. \tag{16}
\end{align*}
$$

The same source justifies analogy between the triples ,event from a field of events interval subjective probability - point estimate of probability" and „object from a universe degree of membership to an intuitionistic fuzzy set [Atanassov, 1999] - degree of membership to a (classical) fuzzy set". On that basis, interval probabilities transform into point estimates using operators that transform intuitionistic degrees of membership into classical degrees of membership [Atanassov, 1988; Atanassov, 1989]. Then the uncertainty interval of the probability of a random event $\theta$ transforms as follows:

$$
\begin{align*}
& \square \theta=<\theta, P_{d}(\theta), 1-P_{d}(\theta)>,  \tag{17}\\
& \diamond \theta=<\theta, P_{u}(\theta), 1-P_{u}(\theta)>,  \tag{18}\\
& D_{\alpha}(\theta)=<x, P_{d}(\theta)+\alpha\left[P_{u}(\theta)-P_{d}(\theta)\right], 1-P_{u}(\theta)+(1-\alpha)\left[P_{u}(\theta)-P_{d}(\theta)\right]>, \\
& \quad \text { for } \alpha \in[0 ; 1], \tag{19}
\end{align*}
$$

where $\square, \diamond$ and $D_{\alpha}$ are the operators necessity, possibility and their fuzzy generalization. Since $D_{0} \equiv \square, D_{1} \equiv \emptyset$, then $D_{\alpha}$ transforms an arbitrary interval probability in the segment ( $\square \theta, \diamond \theta$ ).

Fuzzy-rational lotteries cannot be ranked by expected utility, since (11) does not hold. On the other hand, ranking lotteries of the type (16) is problem under risk and under strict uncertainty, since part of the uncertainty is quantified by subjective probability, and the rest cannot be quantified by the DM. The works [Nikolova, 2006; Tenekedjiev, 2006] propose to transform interval probabilities into point estimate using Wald and Hurwicz $\alpha_{\alpha}$ criteria. The resulting lotteries are classical risky and are ranked according to Wald and Hurwicz ${ }_{\alpha}$ expected utility.

Wald's criterion assumes to increase the probabilities of prizes from their lowest limit $P_{d}\left(\theta_{i, j}\right)$ (but not higher than their upper limit), initiating with the worst outcome, until the corrected probabilities sum to one. Then Wald lotteries are constructed:

$$
l_{i}^{W}=<\square \theta_{i, 1}, x_{1} ; \ldots ; \square \theta_{i, j_{W}^{(i)}-1}, x_{j_{W}^{(i)}-1} ; D_{\beta^{(i)}}\left(\theta_{i, j_{W}^{(i)}}\right), x_{j_{W}^{(i)}} ; \diamond \theta_{i, j_{W}^{(i)}+1}, x_{j_{W}^{(i)}+1} ; \ldots ; \diamond \theta_{i, r}, x_{r}>,(20)
$$

where $j_{W}^{(i)}$ and $\beta^{(i)}$ are assessed using $\beta_{k}^{(i)}$, for $k=1,2, \ldots, r$ :

$$
j_{W}^{(i)}=\arg \left\{\beta_{k}^{(i)} \in(0 ; 1]\right\}, \beta^{(i)}=\beta_{\substack{j_{W}^{(i)}}}^{(i)}
$$

Regardless of the fact that (20) is a fuzzy rational lottery, uncertainty is completely quantified and the lottery may be structured as a classical risky one:

$$
\begin{align*}
& l_{i}^{W c r}=<P^{W}\left(\theta_{i, 1}\right), x_{l} ; P^{W}\left(\theta_{i, 2}\right), x_{2} ; \ldots ; P^{W}\left(\theta_{i, r}\right), x_{r}>,  \tag{23}\\
& P^{W}\left(\theta_{i, j}\right)= \begin{cases}P_{d}\left(\theta_{i, j}\right) & \text { for } j<j_{W}^{(i)} \\
P_{d}\left(\theta_{i, j}\right)+\beta^{(i)}\left[P_{u}\left(\theta_{i, j}\right)-P_{d}\left(\theta_{i, j}\right)\right] & \text { for } j=j_{W}^{(i)} \\
P_{u}\left(\theta_{i, j}\right) & \text { for } j>j_{W}^{(i)}\end{cases} \tag{24}
\end{align*}
$$

Expected utility (25) may be used to rank lotteries (23), which is the Wald expected utility criterion to rank fuzzy-rational lotteries:

$$
\begin{equation*}
E_{i}^{W}(u / p)=\sum_{j=1}^{r} P^{W}\left(\theta_{i, j}\right) u\left(x_{j}\right) . \tag{25}
\end{equation*}
$$

If the maximax criterion is used, then probabilities of prizes are decreased from their upper margin $P_{u}\left(\theta_{i, j}\right)$ (but not lower than their lower margin), initiating with the best prize, until the corrected probabilities sum to one. The anti-Wald lottery is constructed:

$$
\begin{equation*}
l_{i}^{\neg W}=<\diamond \theta_{i, 1}, x_{1} ; \ldots ; \diamond \theta_{i, j_{\neg W}^{(i)-1}}, x_{j_{\neg W}^{(i)-1}} ; D_{\gamma}\left(\theta_{i, j_{\neg W}^{(i)}}\right), x_{j_{\neg W}^{(i)}} ; \square \theta_{i, j_{\neg W}^{(i)+1}}, x_{j_{\sim W}^{(i)}+1} ; \ldots ; \square \theta_{i, r}, x_{r}>, \tag{26}
\end{equation*}
$$

where $j_{\neg W}^{(i)}$ and $\gamma^{(i)}$ are assessed using $\gamma_{k}^{(i)}$ for $k=1,2, \ldots, r$ :

$$
\begin{align*}
& \gamma_{k}^{(i)}= \begin{cases}\frac{1-\sum_{j=k}^{r} P_{d}\left(\theta_{i, j}\right)-\sum_{j=1}^{k-1} P_{u}\left(\theta_{i, j}\right)}{P_{u}\left(\theta_{i, k}\right)-P_{d}\left(\theta_{i, j}\right)}, & \text { for } P_{u}\left(\theta_{i, k}\right)>P_{d}\left(\theta_{i, k}\right) \\
0, & \text { for } P_{u}\left(\theta_{i, k}\right)=P_{d}\left(\theta_{i, k}\right) \text { and }\left(\sum_{k=1}^{r} P_{u}\left(\theta_{i, k}\right)>\sum_{k=1}^{r} P_{d}\left(\theta_{i, k}\right) \text { or } k>1\right) \\
1, \quad \text { for } P_{u}\left(\theta_{i, k}\right)=P_{d}\left(\theta_{i, k}\right) \text { and } \sum_{k=1}^{r} P_{u}\left(\theta_{i, k}\right)=\sum_{k=1}^{r} P_{d}\left(\theta_{i, k}\right) \text { and } k=1\end{cases}  \tag{27}\\
& j_{\neg W}^{(i)}=\arg \left\{\gamma_{k}^{(i)} \in(0 ; 1]\right\}, \gamma^{(i)}=\gamma_{\substack{(i) \\
j_{J W}}}^{(i)} .
\end{align*}
$$

Regardless of the fact that (26) is a fuzzy rational lottery, uncertainty is completely quantified and the lottery may be structured as a classical risky one:

$$
\begin{align*}
& l_{i}^{\neg W}=<P^{\neg W}\left(\theta_{i, 1}\right), x_{1} ; P^{\neg W}\left(\theta_{i, 2}\right), x_{2} ; \ldots ; P^{\neg W}\left(\theta_{i, r}\right), x_{r}>,  \tag{29}\\
& P^{\neg W}\left(\theta_{i, j}\right)= \begin{cases}P_{u}\left(\theta_{i, j}\right), & \text { for } j<j_{\neg W}^{(i)} \\
P_{d}\left(\theta_{i, j}\right)+\gamma^{(i)}\left[P_{u}\left(\theta_{i, j}\right)-P_{d}\left(\theta_{i, j}\right)\right], & \text { for } j=j_{\neg W}^{(i)} \\
P_{d}\left(\theta_{i, j}\right), & \text { for } j>j_{\neg W}^{(i)}\end{cases} \tag{30}
\end{align*}
$$

Expected utility (31) may be used to rank lotteries (29), which is the anti-Wald expected utility criterion to rank fuzzy-rational lotteries:

$$
\begin{equation*}
E_{i}^{\neg W}(u / p)=\sum_{j=1}^{r} P^{\neg W}\left(\theta_{i, j}\right) u\left(x_{j}\right) . \tag{31}
\end{equation*}
$$

If the Hurwicz ${ }_{\alpha}$ criterion is used, the Hurwicz ${ }_{\alpha}$ lottery is constructed, where Wald and anti-Wald probabilities are weighted by $\alpha$ :

$$
\begin{align*}
& l_{i}^{H_{\alpha} c r}=<P^{H_{\alpha}}\left(\theta_{i, 1}\right), x_{1} ; P^{H_{\alpha}}\left(\theta_{i, 2}\right), x_{2} ; \ldots ; P^{H_{\alpha}}\left(\theta_{i, r}\right), x_{r}>,  \tag{32}\\
& P^{H_{\alpha}}\left(\theta_{i, j}\right)=\alpha P^{W}\left(\theta_{i, j}\right)+(1-\alpha) P^{-W}\left(\theta_{i, j}\right) . \tag{33}
\end{align*}
$$

Expected utility (34) may be used to rank lotteries (32), which is the Hurwicz ${ }_{\alpha}$ expected utility criterion to rank fuzzy-rational lotteries:

$$
\begin{equation*}
E_{i}^{H_{\alpha}}(u / p)=\sum_{j=1}^{r} P^{H_{\alpha}}\left(\theta_{i, j}\right) u\left(x_{j}\right) \tag{34}
\end{equation*}
$$

## 5. Formalization of the Ellsberg paradox

### 5.1. Setup

Let $x_{1}$ be the prize of $\$ 1000, x_{2}$ be the prize of $\$ 0$, and the random events $\theta_{1}, \theta_{2}$ and $\theta_{3}$ are respectively ,to draw red/blue/yellow ball from an urn of 90 balls, of which exactly 30 are red". The gambles $A, B, C$ and $D$ may be represented as ordinary lotteries:

$$
\begin{align*}
& l_{A}=\left\langle\theta_{A, 1}, x_{1} ; \theta_{A, 2}, x_{2}>,\right.  \tag{35}\\
& l_{B}=\left\langle\theta_{B, 1}, x_{1} ; \theta_{B, 2}, x_{2}>,\right.  \tag{36}\\
& l_{C}=\left\langle\theta_{C, 1}, x_{1} ; \theta_{C, 2}, x_{2}>,\right.  \tag{37}\\
& l_{D}=<\theta_{D, 1}, x_{1} ; \theta_{D, 2}, x_{2}>, \tag{38}
\end{align*}
$$

where

$$
\left\lvert\, \begin{align*}
& \theta_{A, 1} \equiv \theta_{1} ; \theta_{A, 2} \equiv \theta_{2} \cup \theta_{3} \equiv \bar{\theta}_{1} ;  \tag{39}\\
& \theta_{B, 1} \equiv \theta_{2} ; \theta_{B, 2} \equiv \theta_{1} \cup \theta_{3} \equiv \bar{\theta}_{2} ; \\
& \theta_{C, 1} \equiv \theta_{1} \cup \theta_{3} \equiv \bar{\theta}_{2} ; \theta_{C, 2} \equiv \theta_{2} ; \\
& \theta_{D, 1} \equiv \theta_{2} \cup \theta_{3} \equiv \bar{\theta}_{1} ; \theta_{D, 2} \equiv \theta_{1} .
\end{align*}\right.
$$

The task is to compare according to preference the lotteries in the pairs $l_{A}-l_{B}, l_{C}-l_{D}$.

### 5.2. Solution using classical risky lotteries

According to each rational DM:
a) $x_{1}$ and $x_{2}$ are the most and the least preferred prize, thus $u_{1}=u\left(x_{1}\right)=1, u_{2}=u\left(x_{2}\right)=0$;
b) the probability to draw a red ball is $P\left(\theta_{1}\right)=30 / 90=1 / 3$ and then the probability not to draw a red ball is $P\left(\bar{\theta}_{1}\right)=1-P\left(\theta_{1}\right)=1-1 / 3=2 / 3$;
c) the subjective probability to draw a blue ball is an unique value not greater than $1 / 3$ : $P\left(\theta_{2}\right)=a \in[0 ; 2 / 3]$. Then the subjective probability not to draw a blue ball is $P\left(\bar{\theta}_{2}\right)=1-$ $P\left(\theta_{2}\right)=1-a$;

According to (39), the lotteries (35)-(38) are presented as classical risky ones:

$$
\begin{align*}
& l_{A}^{c r}=<P\left(\theta_{A, 1}\right), x_{1} ; P\left(\theta_{A, 2}\right), x_{2}>=<1 / 3, x_{1} ; 2 / 3, x_{2}>,  \tag{40}\\
& l_{B}^{c r}=<P\left(\theta_{B, 1}\right), x_{1} ; P\left(\theta_{B, 2}\right), x_{2}>=<a, x_{1} ; 1-a, x_{2}>,  \tag{41}\\
& l_{C}^{c r}=<P\left(\theta_{C, 1}\right), x_{1} ; P\left(\theta_{C, 2}\right), x_{2}>=<1-a, x_{1} ; a, x_{2}>,  \tag{42}\\
& l_{D}^{c r}=<P\left(\theta_{D, 1}\right), x_{1} ; P\left(\theta_{D, 2}\right), x_{2} \gg=<2 / 3, x_{1} ; 1 / 3, x_{2}>. \tag{43}
\end{align*}
$$

The expected utilities of (40)-(43) are

$$
\begin{align*}
& E_{A}(u / p)=1 / 3 u_{1}+2 / 3 u_{2}=1 / 3,  \tag{44}\\
& E_{B}(u / p)=a u_{1}+(1-a) u_{2}=a,  \tag{45}\\
& E_{C}(u / p)=(1-a) u_{1}+a u_{2}=1-a,  \tag{46}\\
& E_{D}(u / p)=2 / 3 u_{1}+1 / 3 u_{2}=2 / 3 \tag{47}
\end{align*}
$$

Let the DM prefers $A$ to $B$ and then

$$
\begin{equation*}
E_{A}(u / p)>E_{B}(u / p)=>1 / 3>a . \tag{48}
\end{equation*}
$$

Then $(-1 / 3)<(-a)$, which means that $2 / 3=1-1 / 3<(1-a)$, and it follows that

$$
\begin{equation*}
E_{C}(u / p)=1-a>2 / 3=E_{D}(u / p) \tag{49}
\end{equation*}
$$

From (49) it follows that the DM should prefer $C$ to $D$. It follows that people that prefer $A$ to $B$ and $C$ to $D$, are not rational.

### 5.3. Solution using fuzzy-rational lotteries

According to each fuzzy-rational DM:
a) $x_{1}$ and $x_{2}$ are the most and least preferred prizes, thus $u\left(x_{1}\right)=1, u_{2}=u\left(x_{2}\right)=0$;
b) the probability to draw a red ball is $P\left(\theta_{1}\right)=30 / 90=1 / 3$, thus the probability not to draw a red ball is $P\left(\overline{\theta_{1}}\right)=1-P\left(\theta_{1}\right)=1-1 / 3=2 / 3$;
c) the subjective probability to draw a blue ball $P\left(\theta_{2}\right) \in[0 ; 2 / 3]$;
d) the subjective probability to draw a yellow ball $P\left(\theta_{3}\right) \in[0 ; 2 / 3]$;
e) the subjective probability not to draw a blue ball equals the sum of the subjective probability to draw a red ball and of the subjective probability to draw a yellow ball: $P\left(\bar{\theta}_{2}\right)=P\left(\theta_{1} \cup \theta_{3}\right)=P\left(\theta_{1}\right)+P\left(\theta_{3}\right) \in[1 / 3 ; 1]$.

According to (39), the lotteries (35)-(38) may be presented as fuzzy-rational lotteries:

$$
\begin{align*}
l_{A}^{f r} & =\ll \theta_{A, 1}, P_{d}\left(\theta_{A, 1}\right), 1-P_{u}\left(\theta_{A, 1}\right)>, x_{1} ;<\theta_{A, 2}, P_{d}\left(\theta_{A, 2}\right), 1-P_{u}\left(\theta_{A, 2}\right)>, x_{2}>= \\
& =\ll \theta_{A, 1}, 0,1>, x_{1} ;<\theta_{A, 2}, 1 / 3,0>, x_{2}>  \tag{50}\\
l_{B}^{f r} & =\ll \theta_{B, 1}, P_{d}\left(\theta_{B, 1}\right), 1-P_{u}\left(\theta_{B, 1}\right)>, x_{1} ;<\theta_{B, 2}, P_{d}\left(\theta_{B, 2}\right), 1-P_{u}\left(\theta_{B, 2}\right)>, x_{2}>= \\
& =\ll \theta_{B, 1}, 0,1>, x_{1} ;<\theta_{B, 2}, 1 / 3,0>, x_{2}>  \tag{51}\\
l_{C}^{f r} & =\ll \theta_{C, 1}, P_{d}\left(\theta_{C, 1}\right), 1-P_{u}\left(\theta_{C, 1}\right)>, x_{1} ;<\theta_{C, 2}, P_{d}\left(\theta_{C, 2}\right), 1-P_{u}\left(\theta_{C, 2}\right)>, x_{2}>= \\
& =\ll \theta_{C, 1}, 1 / 3,0>, x_{1} ;<\theta_{C, 2}, 0,1 / 3>, x_{2}>  \tag{52}\\
l_{D}^{f r} & =\ll \theta_{D, 1}, P_{d}\left(\theta_{D, 1}\right), 1-P_{u}\left(\theta_{D, 1}\right)>, x_{1} ;<\theta_{D, 2}, P_{d}\left(\theta_{D, 2}\right), 1-P_{u}\left(\theta_{D, 2}\right)>, x_{2}>= \\
& =\ll \theta_{D, 1}, 1 / 3,0>, x_{1} ;<\theta_{D, 2}, 0,1 / 3>, x_{2}>. \tag{53}
\end{align*}
$$

If Wald expected utility criterion is used, then according to (21) and (22) it follows that $\beta_{1}^{(B)}=0, \quad \beta_{2}^{(B)}=1, \quad j_{W}^{(B)}=2, \quad \beta^{(B)}=1, \quad P^{W}\left(\theta_{B, 1}\right)=P_{d}\left(\theta_{B, 1}\right)=0, \quad P^{W}\left(\theta_{B, 2}\right)=P_{d}\left(\theta_{B, 2}\right)+$ $+\beta^{(B)}\left[P_{u}\left(\theta_{B, 2}\right)-P_{d}\left(\theta_{B, 2}\right)\right]=1$ and (51) transforms into the Wald lottery

$$
\begin{equation*}
l_{B}^{W c r}=<P^{W}\left(\theta_{B, 1}\right), x_{1} ; P^{W}\left(\theta_{B, 2}\right), x_{2}>=<0, x_{1} ; 1, x_{2}>. \tag{54}
\end{equation*}
$$

By analogy, $\beta_{1}^{(C)}=0, \beta_{2}^{(C)}=1, j_{W}^{(C)}=2, \beta^{(C)}=1, P^{W}\left(\theta_{C, 1}\right)=P_{d}\left(\theta_{C, 1}\right)=1 / 3, P^{W}\left(\theta_{C, 2}\right)=$ $=P_{d}\left(\theta_{C, 2}\right)+\beta^{(B)}\left[P_{u}\left(\theta_{C, 2}\right)-P_{d}\left(\theta_{C, 2}\right)\right]=2 / 3$ and (52) transforms into the Wald lottery

$$
\begin{equation*}
l_{C}^{W c r}=<P^{W}\left(\theta_{C, 1}\right), x_{1} ; P^{W}\left(\theta_{C, 2}\right), x_{2}>=<1 / 3, x_{1} ; 2 / 3, x_{2}>. \tag{55}
\end{equation*}
$$

If the anti-Wald criterion is used, then according to (27) and (28) it follows that $\gamma_{1}^{(B)}=1$, $\gamma_{2}^{(B)}=0, \quad j_{\neg W}^{(B)}=1, \quad \gamma^{(B)}=1, \quad P^{\urcorner W}\left(\theta_{B, 1}\right)=P_{d}\left(\theta_{B, 1}\right)+\gamma^{(B)}\left[P_{u}\left(\theta_{B, 1}\right)-P_{d}\left(\theta_{B, 1}\right)\right]=2 / 3, \quad P^{W}\left(\theta_{B, 2}\right)=$ $=P_{d}\left(\theta_{B, 2}\right)=1 / 3$ and (51) transforms into the anti-Wald lottery

$$
\begin{equation*}
l_{B}^{-W c r}=<P^{-W}\left(\theta_{B, 1}\right), x_{1} ; P \neg^{W}\left(\theta_{B, 2}\right), x_{2}>=<2 / 3, x_{1} ; 1 / 3, x_{2}>. \tag{56}
\end{equation*}
$$

By analogy, $\quad \gamma_{1}^{(C)}=1, \quad \gamma_{2}^{(C)}=0, \quad j_{\neg W}^{(C)}=1, \quad \gamma^{(C)}=1, \quad P^{\square}\left(\theta_{C, 1}\right)=P_{d}\left(\theta_{C, 1}\right)+$ $+\gamma^{(B)}\left[P_{u}\left(\theta_{C, 1}\right)-P_{d}\left(\theta_{C, 1}\right)\right]=1, P^{-W}\left(\theta_{C, 2}\right)=P_{d}\left(\theta_{C, 2}\right)=0$ and (52) transforms into the anti-Wald lottery

$$
\begin{equation*}
l_{C} \vec{C}^{W c r}=<P^{\neg W}\left(\theta_{C, 1}\right), x_{1} ; P^{-W}\left(\theta_{C, 2}\right), x_{2}>=<1, x_{1} ; 0, x_{2}>. \tag{57}
\end{equation*}
$$

If the Hurwicz ${ }_{\alpha}$ criterion is used at a given value of $\alpha$, then according to (33) for the lottery $\quad(51), \quad P^{H_{\alpha}}\left(\theta_{B, 1}\right)=\alpha P^{W}\left(\theta_{B, 1}\right)+(1-\alpha) P^{\neg^{W}}\left(\theta_{B, 1}\right)=2 / 3-2 / 3 \alpha, \quad P^{H_{\alpha}}\left(\theta_{B, 2}\right)=$ $=\alpha P^{W}\left(\theta_{B, 2}\right)+(1-\alpha) P^{-W}\left(\theta_{B, 2}\right)=1 / 3+2 / 3 \alpha$. Then (51) transforms into the Hurwicz ${ }_{\alpha}$ lottery

$$
\begin{equation*}
l_{B}^{H_{\alpha} c r}=<P^{H_{\alpha}}\left(\theta_{B, 1}\right), x_{1} ; P^{H_{\alpha}}\left(\theta_{B, 2}\right), x_{2}>=<2 / 3-2 / 3 \alpha, x_{1} ; 1 / 3+2 / 3 \alpha, x_{2}>. \tag{58}
\end{equation*}
$$

By analogy, for (52) $\quad P^{H_{\alpha}}\left(\theta_{C, 1}\right)=\alpha P^{W}\left(\theta_{C, 1}\right)+(1-\alpha) P^{\neg^{W}}\left(\theta_{C, 1}\right)=1-2 / 3 \alpha$, $P^{H_{\alpha}}\left(\theta_{C, 2}\right)==\alpha P^{W}\left(\theta_{C, 2}\right)+(1-\alpha) P^{W}\left(\theta_{C, 2}\right)=2 / 3 \alpha$ and (52) transforms into the Hurwicz ${ }_{\alpha}$ lottery

$$
\begin{equation*}
l_{C}^{H_{\alpha} c r}=<P^{H_{\alpha}}\left(\theta_{C, 1}\right), x_{1} ; P^{H_{\alpha}}\left(\theta_{C, 2}\right), x_{2}>=<1-2 / 3 \alpha, x_{1} ; 2 / 3 \alpha, x_{2}>. \tag{59}
\end{equation*}
$$

Regardless of the fact that (50) and (53) are fuzzy rational lotteries, uncertainty is completely quantified and then

$$
\begin{equation*}
l_{A}^{H_{\alpha} c r}=l_{A}^{W c r}=l_{A}^{W c r}=l_{A}^{c r}=<P\left(\theta_{A, 1}\right), x_{1} ; P\left(\theta_{A, 2}\right), x_{2}>=<1 / 3, x_{1} ; 2 / 3, x_{2}>, \tag{60}
\end{equation*}
$$

$$
\begin{equation*}
l_{D}^{H_{\alpha} c r}=l_{D}^{W c r}=l_{D}^{W c r}=l_{D}^{c r}=<P\left(\theta_{D, 1}\right), x_{1} ; P\left(\theta_{D, 2}\right), x_{2}>=<2 / 3, x_{1} ; 1 / 3, x_{2}>. \tag{61}
\end{equation*}
$$

The expected utilities of (58) - (61) may be found using (34):

$$
\begin{align*}
& E_{A}^{H_{\alpha}}(u / p)=1 / 3 u_{1}+2 / 3 u_{2}=1 / 3,  \tag{62}\\
& E_{B}^{H_{\alpha}}(u / p)=(2 / 3-2 / 3 \alpha) u_{1}+(1 / 3+2 / 3 \alpha) u_{2}=2 / 3-2 / 3 \alpha,  \tag{63}\\
& E_{C}^{H_{\alpha}}(u / p)=(1-2 / 3 \alpha) u_{1}+2 / 3 \alpha u_{2}=1-2 / 3 \alpha,  \tag{64}\\
& E_{D}^{H_{\alpha}}(u / p)=2 / 3 u_{1}+1 / 3 u_{2}=2 / 3 . \tag{65}
\end{align*}
$$

Let the DM prefers $A$ to $B$ and then

$$
\begin{equation*}
E_{A}^{H_{\alpha}}(u / p)>E_{B}^{H_{\alpha}}(u / p)=>1 / 3>2 / 3-2 / 3 \alpha=>\alpha>1 / 2 \tag{66}
\end{equation*}
$$

Then $(-2 / 3 \alpha)<(-1 / 3)$, thus $1-2 / 3 \alpha<1-1 / 3=2 / 3$, and it follows that

$$
\begin{equation*}
E_{D}^{H_{\alpha}}(u / p)=2 / 3>1-2 / 3 \alpha=E_{C}^{H_{\alpha}}(u / p) . \tag{67}
\end{equation*}
$$

From (67) it follows that the DM must prefer $D$ to $C$. Thus, people who prefer $A$ to $B$ and $D$ to $C$, make a consistent choice, in case they are moderate or extreme pessimists, since $\alpha>1 / 2$. Regardless of Hurwicz's initial idea, it is not a problem for the DM to elicit $\alpha$ in each problem. It is assumed in [Rapoport, 1989], on the basis of empirical evidences in [Cohen, Jaffray, Said, 1985] that in situation of expected profit people prefer to be pessimists and $\alpha$ is greater than $1 / 2$. The Ellsberg paradox is a task with expected profit and thus it is not surprising that most people, acting as pessimists, intuitively choose $A$ to $B$ and $D$ to $C$. Since this is also the recommended choice of the $H_{u r w i c z}^{\alpha}$ expected utility, then there is actually no paradox.

## Conclusion

Obviously, interval estimates of probabilities should not be related to the DM's value system, i.e. her/his preferences over prizes should not change her/his expectations to receive them. However, if the Ellsberg paradox is analyzed from the point of view of the rational DM, then it is assumed without argumentation that finding point estimates is also not related to the value system of the DM . On the contrary, point estimates of probabilities in the Hurwicz ${ }_{\alpha}$ lotteries strongly rely on the fact that prizes are ordered according to preferences, and account for pessimism/optimism of DMs.

The Hurwicz ${ }_{\alpha}$ expected utility criterion may also justify ambiguity. Wide probability uncertainty intervals shall be identified for events that the DM feels incompetent about, and
vise versa. Assume that the DM must choose between the gambles $l_{1}=\left\langle P\left(\theta_{1}\right), x_{1} ; P\left(\bar{\theta}_{1}\right), x_{2}\right\rangle$ and $l_{2}=<P\left(\theta_{2}\right), x_{1} ; P\left(\bar{\theta}_{2}\right), x_{2}>$, where $\theta_{1}$ is a random event that the DM feels competent about, and $\theta_{2}$ is an event that the DM feels incompetent about. Assume also that the DM has elicited the probabilities of both events as $P\left(\theta_{1}\right) \in\left[P_{d}\left(\theta_{1}\right) ; P_{u}\left(\theta_{1}\right)\right]$ and $P\left(\theta_{2}\right) \in\left[P_{d}\left(\theta_{2}\right) ; P_{u}\left(\theta_{2}\right)\right]$, where $P_{d}\left(\theta_{1}\right)+P_{u}\left(\theta_{1}\right) \approx P_{d}\left(\theta_{2}\right)+P_{u}\left(\theta_{2}\right)$. Then usually $P_{u}\left(\theta_{2}\right)-P_{d}\left(\theta_{2}\right)>P_{u}\left(\theta_{1}\right)-P_{d}\left(\theta_{1}\right)$ and $P_{d}\left(\theta_{1}\right)>P_{d}\left(\theta_{2}\right)$, since the DM is more confident in her/his knowledge about $\theta_{1}$ than about $\theta_{2}$. Having in mind that most people choose according to the Hurwicz ${ }_{\alpha}$ expected utility criterion at values $\alpha$ close to one, and that $u_{I}=1, u_{2}=0$, then the expected utilities of $l_{I}$ and $l_{2}$ shall be $E_{l}(u / p)=P_{d}\left(\theta_{1}\right), E_{2}(u / p)=P_{d}\left(\theta_{2}\right)$. Then ambiguity seems quite rational, since it recommends the choice of the gamble $l_{1}$ that the DM understands well than the gamble $l_{2}$ that is unclear for the DM. Due to the interval estimates of subjective probabilities, human behavior does not seem to be as irrational as normative decision schemes claim it to be.

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