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IFS-valued possibility and necessity measures

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Abstract. In this paper we study IF-valued possibility and necessity measures and IF-valued belief and plausibility measures and some properties of the induced mappings.

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1 Introduction

In the paper [3] there are defined IF-valued fuzzy measures and some properties of duality, continuity and decomposability of IF-valued fuzzy measures. The author shown that every superadditive and subadditive measure is IF-valued fuzzy measure. He also presented some examples.

In *Section* 2 we introduce IF-valued fuzzy measure. IF-valued possibility and necessity measures are studied in *Section* 3 and IF-valued belief and plausibility measures are studied in *Section* 4. We also study some properties of the induced mappings in these sections.

2 IF-valued fuzzy measures

Consider the set L^*

$$L^* = \{(x_1, y_1) \mid (x_1, y_1) \in [0,1]^2 \text{ and } x_1 + y_1 \leq 1\}$$

and order relation $\leq_{L^*}$ defined by

$$(x_1, y_1) \leq_{L^*} (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \geq y_2 \quad \forall (x_1, y_1), (x_2, y_2) \in L^*$$

Then $(L^*, \leq_{L^*})$ is a complete lattice. We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. We also denote

$$\begin{aligned}
\sup_{L^*} A &= (\sup\{x \in [0,1] \mid \exists y \in [0,1] : (x, y) \in A\}, \\
&\quad \inf\{y \in [0,1] \mid \exists x \in [0,1] : (x, y) \in A\}) \\
\inf_{L^*} A &= (\inf\{x \in [0,1] \mid \exists y \in [0,1] : (x, y) \in A\}, \\
&\quad \sup\{y \in [0,1] \mid \exists x \in [0,1] : (x, y) \in A\})
\end{aligned}$$

In the paper $(X, \mathcal{A})$ will denote a measurable space.

Definition 2.1 *A map* $v : \mathcal{A} \to L^*$ *with following properties:*
(1) $v(\emptyset) = (0, 1)$;
(2) $v(X) = (1, 0)$;
(3) $A \subseteq B$ *implies* $v(A) \leq_{L^*} v(B)$
is called an IF-valued fuzzy measure.

3 Possibility and necessity measure

Definition 3.1 *A set function* $\pi : \mathcal{A} \to [0, 1]$ *is called a possibility measure if* π *satisfies following conditions:*
(1) $\pi(\emptyset) = 0$;
(2) $\pi(X) = 1$;
(3) $\pi(\cup_{i\in I} E_i) = \sup_{i\in I} \pi(E_i)$
where $\{E_i\}_{i\in I} \in \mathcal{A}$, $\cup_{i\in I} E_i \in \mathcal{A}$

Definition 3.2 *A set function* $m : \mathcal{A} \to [0, 1]$ *is called a necessity measure if* m *satisfies following conditions:*
(1) $m(\emptyset) = 0$;
(2) $m(X) = 1$;
(3) $m(\cap_{i\in I} E_i) = \inf_{i\in I} m(E_i)$
where $\{E_i\}_{i\in I} \in \mathcal{A}$, $\cap_{i\in I} E_i \in \mathcal{A}$

For every possibility measure π we can obtain a corresponding necessity measure m by

$$m(A) = 1 - \pi(A^c)$$

for $A \in \mathcal{A}$, where $A^c = X \backslash A$.

Definition 3.3 *A set function* $\mu : \mathcal{A} \to [0, 1]$ *which satisfies:*
(1) $\mu(\emptyset) = 0$;
(2) $\mu(X) = 1$;
(3) $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$
is called a fuzzy measure.

Proposition 3.1 *If* $\pi : \mathcal{A} \to [0, 1]$ *is a possibility measure, then* π *is fuzzy measure.*

Proof.
(1) $\pi(\emptyset) = 0$;
(2) $\pi(X) = 1$;
(3) *Let* $A, B \in \mathcal{A}$; $A \subseteq B$, then

$$\pi(B) = \pi(A \cup (B\backslash A)) = \pi(A) \vee \pi(B\backslash A) \geq \pi(A)$$

Proposition 3.2 *If* $m : \mathcal{A} \to [0, 1]$ *is a necessity measure, then* m *is fuzzy measure.*

Proof.
(1) $m(\emptyset) = 0$;
(2) $m(X) = 1$;
(3) *Let* $A, B \in \mathcal{A}$; $A \subseteq B$, then

$$m(A) = m(A \cap B) = m(A) \wedge m(B) \leq m(B)$$

Theorem 3.4 *Let $\pi : \mathcal{A} \to [0,1]$ be a possibility measure. Then $v_\pi : \mathcal{A} \to L^*$ defined by $v_\pi(A) = (\pi(A), 1 - \pi(A))$ is an IF-valued fuzzy measure.*

Proof. We have:
(1) $v_\pi(\emptyset) = (\pi(\emptyset), 1 - \pi(\emptyset)) = (0,1)$;
(2) $v_\pi(X) = (\pi(X), 1 - \pi(X)) = (1,0)$;
(3) $If A \subseteq B$ then $\pi(A) \le \pi(B)$ and $1 - \pi(A) \ge 1 - \pi(B)$. We obtain $v_\pi(A) \le_{L^*} v_\pi(B)$.

Theorem 3.5 *Let $m : \mathcal{A} \to [0,1]$ be a necessity measure. Then $v_m : \mathcal{A} \to L^*$ defined by $v_m(A) = (m(A), 1 - m(A))$ is an IF-valued fuzzy measure.*

Proof. We have:
(1) $v_m(\emptyset) = (m(\emptyset), 1 - m(\emptyset)) = (0,1)$;
(2) $v_m(X) = (m(X), 1 - m(X)) = (1,0)$;
(3) $If A \subseteq B$ then $m(A) \le m(B)$ and $1 - m(A) \ge 1 - m(B)$. We obtain $v_m(A) \le_{L^*} v_m(B)$.

For every possibility measure π there exist corresponding necessity measure m such that $m(A) = 1 - \pi(A^c)$. For $A \in \mathcal{A}$ we can also write:

$$v_\pi(A) \;=\; (\pi(A), m(A^c)) \;=\; (1 - m(A^c), 1 - \pi(A))$$

Theorem 3.6 *If $\pi : \mathcal{A} \to [0,1]$ is a possibility measure, v_π is an IF-valued fuzzy measure induced by π, then v_π satisfies the conditions of possibility measure.*

Proof. We have
(1) $v_\pi(\emptyset) = (0,1)$ - the least element of L^*
(2) $v_\pi(x) = (1,0)$ - the greatest element of L^*
(3) $if \{E_i\}_{i \in I} \in \mathcal{A},\ \cup_{i \in I} E_i \in \mathcal{A}$, then

$$v_\pi(\cup_{i \in I} E_i) = (\pi(\cup_{i \in I} E_i), 1 - \pi(\cup_{i \in I} E_i)) =$$

$$= (\sup_{i \in I} \pi(E_i), 1 - \sup_{i \in I} \pi(E_i)) = (\sup_{i \in I} \pi(E_i), \inf_{i \in I}(1 - \pi(E_i))) =$$

$$= \sup_{i \in I}(\pi(E_i), 1 - \pi(E_i)) = \sup_{i \in I} v_\pi(E_i)$$

Theorem 3.7 *If $m : \mathcal{A} \to [0,1]$ is a necessity measure, v_m is an IF-valued fuzzy measure induced by m, then v_m satisfies the conditions of necessity measure.*

Proof. We have
(1) $v_m(\emptyset) = (0,1)$ - the least element of L^*
(2) $v_m(x) = (1,0)$ - the greatest element of L^*
(3) $if \{E_i\}_{i \in I} \in \mathcal{A},\ \cup_{i \in I} E_i \in \mathcal{A}$, then

$$v_m(\cap_{i \in I} E_i) = (m(\cap_{i \in I} E_i), 1 - m(\cap_{i \in I} E_i)) =$$

$$= (\inf_{i \in I} m(E_i), 1 - \inf_{i \in I} m(E_i)) = (\inf_{i \in I} m(E_i), \sup_{i \in I}(1 - m(E_i))) =$$

$$= \inf_{i \in I}(m(E_i), 1 - m(E_i)) = \inf_{i \in I} v_m(E_i)$$

4 Belief and plausibility measure

Definition 4.1 *A set function $m : \mathcal{A} \to [0,1]$ is a basic probability assignment if it satisfies the conditions:*
(1) $m(\emptyset) = 0$;
(2) $\sum_{A \in \mathcal{A}} m(A) = 1$.

Definition 4.2 *Let m be a probability assignment on $\mathcal{A}$. The belief measure $Bel : \mathcal{A} \to [0,1]$ induced by m is defined by:*

$$Bel(A) = \sum_{B \subset A} m(B) \quad (A \in \mathcal{A}).$$

Belief measure satisfies the following conditions:
(1) $Bel(\emptyset) = 0$;
(2) $Bel(X) = 1$;
(3) $Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{i=1}^{n} Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + \ldots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \ldots \cap A_n)$
for $A_1, A_2, \ldots, A_n \in \mathcal{A}$.
Hence belief measure is monotone and superadditive.

Definition 4.3 *Let m be a probability assignment on $\mathcal{A}$. Plausibility measure $Pl : \mathcal{A} \to [0,1]$ induced by m is defined by:*

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \quad (A \in \mathcal{A}).$$

If Bel and Pl are induced by the same probability assignment then

$$Bel(A) = 1 - Pl(A^c) \quad \textit{and} \quad Bel(A) \leq Pl(A)$$

This implies that the plausibility measure satisfies the following condition: $Pl(A_1 \cap A_2 \cap \ldots \cap A_n) \leq \sum_{i=1}^{n} Pl(A_i) - \sum_{i<j} Pl(A_i \cup A_j) + \ldots + (-1)^{n+1} Pl(A_1 \cup A_2 \cup \ldots \cup A_n)$
for $A_1, A_2, \ldots, A_n \in \mathcal{A}$.
Hence plausibility measure is monotone and subadditive.

Proposition 4.1 *Belief measure and plausibility measure are fuzzy measures.*

Proof. For belief measure we have:
(1) $Bel(\emptyset) = 0$;
(2) $Bel(X) = 1$;
(3) *Let* $A, B \in \mathcal{A};\ A \subseteq B$,
because belief measure is superadditive measure we obtain

$$Bel(B) = Bel(A \cup B) \geq Bel(A) + Bel(B) \geq Bel(A)$$

For plausibility measure we have:
(1) $Pl(\emptyset) = 0$;
(2) $Pl(X) = 1$;
(3) *Let* $A, B \in \mathcal{A};\ A \subseteq B$, than $B^c \subseteq A^c$.
Because $Pl(A) = 1 - Bel(A^c)$ we obtain

$$Pl(B) = 1 - Bel(B^c) \geq 1 - Bel(A^c) = Pl(A)$$

Theorem 4.4 *Let Bel be a belief measure and Pl be a plausibility measure. Then* $v(A) = (Bel(A), 1 - Pl(A))$ *is an IF-valued fuzzy measure.*

Proof. We have
(1) $v(\emptyset) = (Bel(\emptyset), 1 - Pl(\emptyset)) = (0, 1)$;
(2) $v(X) = (Bel(X), 1 - Pl(X)) = (1, 0)$;
(3) $If A \subseteq B$, than $Bel(A) \leq Bel(B)$ and $1 - Pl(A) \geq 1 - Pl(B)$
We obtain $v(A) \leq_{L^*} v(B)$

Theorem 4.5 *Let Bel be a belief measure and Pl be a plausibility measure. Then* $v(A) = (Pl(A), 1 - Bel(A))$ *is an IF-valued fuzzy measure.*

Proof. We have
(1) $v(\emptyset) = (Pl(\emptyset), 1 - Bel(\emptyset)) = (0, 1)$;
(2) $v(X) = (Pl(X), 1 - Bel(X)) = (1, 0)$;
(3) $If A \subseteq B$, than $Pl(A) \leq Pl(B)$ and $1 - Bel(A) \geq 1 - Bel(B)$
We obtain $v(A) \leq_{L^*} v(B)$

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