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IF-generators

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Abstract: In the paper we study the notion of the entropy of dynamical systems based on IF-events (Kolmogorov–Sinaj type). We define the notion of the IF-generator and we study some relationships between this notions. We show some possibility how to calculate this entropy.
Keywords: IF-generator, IF-event, IF-partition, Entropy, IF-dynamical system.
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1 Introduction

We start with classical dynamical systems $(\Omega, \mathcal{S}, P, T)$, where $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a measure preserving map, i.e. $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$. The entropy of the dynamical system is defined as follows (see [11]). Consider measurable partition $\mathcal{A} = \{A_1, ..., A_k\}$, where $A_i \in \mathcal{S}; i = 1, ..., k, A_i \cap A_j = \emptyset; i \neq j, \bigcup_{i=1}^{k} A_i = \Omega$. Its entropy is the number

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(P(A_i)),$$

where $\varphi(x) = -x \log x$, if $x > 0$, and $\varphi(0) = 0$. If $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{B} = \{B_1, ..., B_l\}$ are two measurable partitions, then $T^{-1}(\mathcal{A}) = \{T^{-1}(A_1), ..., T^{-1}(A_k)\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}$ are measurable partitions, too. It can be proved that there exists

$$h(\mathcal{A}, T) = \lim_{n\to\infty} H\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right).$$

The entropy $h(T)$ of $(\Omega, \mathcal{S}, P, T)$ is defined as the supremum

$$h(T) = \sup\{h(\mathcal{A}, T); \mathcal{A} \text{ is a measurable partition}\}.$$

The aim of the Kolmogorov–Sinaj entropy was to distinguish non-isomorphic dynamical systems. Two dynamical systems with different entropies cannot be isomorphic.

The notion of the entropy has been extended using fuzzy partitions instead of set partitions (see [7, 11]). Let $\mathcal{T}$ be a tribe of fuzzy sets on Ω. Fuzzy partition is a set of functions $\mathcal{A} = \{f_1, ..., f_k\} \subset \mathcal{T}$ such that $\sum_{i=1}^{k} f_i = 1$. Then we define its entropy

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(m(f_i)) \tag{1}$$

and the conditional entropy

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{k} \sum_{j=1}^{l} m(g_j) \varphi \left(\frac{m(f_i \cdot g_j)}{m(g_j)} \right), \tag{2}$$

where $\mathcal{A} = \{f_1, ..., f_k\}, \mathcal{B} = \{g_1, ..., g_l\}$ are fuzzy partitions. Further

$$h(\mathcal{A}, \tau) = \lim_{n \to \infty} \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}) \right),$$

and, if $G \subset \mathcal{T}$ is an arbitrary non-empty set, then

$$h_G(\tau) = \sup\{h(\mathcal{A}, \tau); \mathcal{A} \text{ is a fuzzy partition, } \mathcal{A} \subset G\}.$$

We extend the notion of the entropy to dynamical systems based on IF-events (see [1, 2, 3, 4, 12]). An IF-event is a pair $A = (\mu_A, \nu_A)$ of $\mathcal{S}$-measurable function $\mu_A, \nu_A : \Omega \to [0, 1]$ such that

$$\mu_A + \nu_A \leq 1.$$

If $f_A : \Omega \to [0, 1]$ is a fuzzy set, then the pair $(f_A, 1 - f_A)$ is an IF-event, of course IF-events present a larger family. Denote by $\mathcal{F}$ the family of all IF-events. On $\mathcal{F}$ we define a partial binary operation $+$ and a binary operation $\cdot$ by the formulas

$$A \oplus B = (\mu_A, \nu_A) \oplus (\mu_B, \nu_B) = (\mu_A + \mu_B, \nu_A + \nu_B - 1),$$
$$\text{whenever } \mu_A + \mu_B \leq 1 \text{ and } 0 \leq \nu_A + \nu_B - 1 \leq 1,$$

and

$$A \odot B = (\mu_A, \nu_A) \odot (\mu_B, \nu_B) = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B).$$

Also, on $\mathcal{F}$ we define partial a unary operation $*$ by the formula

$$A^* = (\mu_A, \nu_A)^* = (1 - \mu_A, 1 - \nu_A),$$

whenever $(1 - \mu_A) + (1 - \nu_A) \leq 1$. Further

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B,$$

where $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \mathcal{F}$. So, the smallest element in family $\mathcal{F}$ is $(0, 1)$ and the biggest one is element $(1, 0)$.

2 IF-dynamical system

Definition 2.1. *By a state on the family $\mathcal{F}$ of all IF-events we mean a mapping $m : \mathcal{F} \to [0,1]$ satisfying following conditions*

(i) $m((1,0)) = 1$;

(ii) If $A, B, C \in \mathcal{F}$ and $A \oplus B = C$, then $m(A) + m(B) = m(C)$;

(iii) If $A_n \in \mathcal{F}(n = 1, 2, ...)$, $A_n \nearrow A$, then $m(A_n) \nearrow m(A)$.

Definition 2.2. *Let $m : \mathcal{F} \to [0,1]$ be a state on the family of all IF-events $\mathcal{F}$ and $\tau : \mathcal{F} \to \mathcal{F}$ be a mapping satisfying following conditions*

(I) If $A \in \mathcal{F}$, then $\tau(A) \in \mathcal{F}$ and $m(A) = m(\tau(A))$.

(II) If $A, B \in \mathcal{F}$ and there exists $A \oplus B$, then $\tau(A + B) = \tau(A) + \tau(B)$.

Then a triplet $(\mathcal{F}, m, \tau)$ is an IF-dynamical system.

To any state on $\mathcal{F}$ there exists $\alpha \in [0,1]$ such that

$$m_\alpha(A) = m(A) = m((\mu_A, \nu_A)) = (1-\alpha) \int_\Omega \mu_A dP + \alpha \int_\Omega (1 - \nu_A) dP. \tag{3}$$

See [8]. Following this result it is reasonable to consider the family $\mathcal{F}$ and a mapping (state) $m_\alpha : \mathcal{F} \to [0,1]$ defined by (3). Finally, let a mapping $\tau : \mathcal{F} \to \mathcal{F}$ be defined by $\tau(A) = \tau((\mu_A, \nu_A)) = (\mu_A \circ T, \nu_A \circ T) = A \circ T$. Then $(\mathcal{F}, m_\alpha, \tau)$ is an IF-dynamical system (see [5]).

3 Entropy of IF-partition

We want to define the entropy of the dynamical system $(\mathcal{F}, m_\alpha, \tau)$. The crucial point in the definition is the notion of an IF-partition. We shall consider a family of all couples of fuzzy sets

$$\mathcal{M} = \{(f,g) : \Omega \to [0,1]^2; f, g \text{ are } \mathcal{S}\text{-measurable}\}.$$

On $\mathcal{M}$ we define a partial binary operation $+$ and a binary operation $\cdot$ by the formulas

$$(f,g) + (h,k) = (f + h, g + k - 1), \text{ whenever } f + h \leq 1 \text{ and } 0 \leq g + k - 1 \leq 1,$$

and

$$(f,g) \cdot (h,k) = (fh, g + k - gk).$$

See [9]. Of course, these partial binary operations are extensions of partial binary operations $\oplus$ and $\odot$ from the family $\mathcal{F}$ of all IF-events to the family M. Recall that operations $+$ and $\cdot$ fulfill the commutative, associative and distributive law, and a couple $(\mathcal{M}, \cdot)$ is an MV-algebra with product (see [10]).

Definition 3.1. *An IF-partition is any set* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\} \subset \mathcal{M}$ *such that*

$$(\mu_{A_1}, \nu_{A_1}) \oplus (\mu_{A_2}, \nu_{A_2}) \oplus ... \oplus (\mu_{A_k}, \nu_{A_k}) = (1, 0).$$

If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ are two IF-partitions, then $\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\}$ and $\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j});\ i = 1, ..., k, j = 1, ..., l\}$ are IF-partitions, too.

The definition of the entropy of the fuzzy partition in the definition of the of the IF-partition is used [5], [6]. If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ is an IF-partition, then $\mathcal{A}^\flat = \{\mu_{A_1}, ..., \mu_{A_k}\}$ and $\mathcal{A}^\sharp = \{1 - \nu_{A_1}, ..., 1 - \nu_{A_k}\}$ are fuzzy partitions.

Definition 3.2. *If* $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ *is an IF-partition, then we define its entropy (with respect to a given state* m_α*)*

$$H_\alpha(\mathcal{A}) = (1 - \alpha)H(\mathcal{A}^\flat) + \alpha H(\mathcal{A}^\sharp),$$

where H *is the entropy of the fuzzy partition (see equation (1)).*

Definition 3.3. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are two IF-partitions, then we define the conditional entropy (with respect to a given state* m_α*)*

$$H_\alpha(\mathcal{A}|\mathcal{B}) = (1 - \alpha)H(\mathcal{A}^\flat|\mathcal{B}^\flat) + \alpha H(\mathcal{A}^\sharp|\mathcal{B}^\sharp),$$

where H *is the conditional entropy of fuzzy partitions (see equation (2)).*

Proposition 3.4. *If* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *are IF-partitions, then the following properties are satisfied:*

(i) If $\mathcal{B} \leq \mathcal{C}$*, then* $H_\alpha(\mathcal{A}|\mathcal{C}) \leq H_\alpha(\mathcal{A}|\mathcal{B})$*;*

(ii) $H_\alpha(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) = H_\alpha(\mathcal{B}|\mathcal{A}) + H_\alpha(\mathcal{C}|\mathcal{B} \vee \mathcal{A})$.

Proof. See [6]. □

4 Entropy on IF-dynamical system

Definition 4.1. *For every IF-partition* $\mathcal{A}$ *we define*

$$h_\alpha(\mathcal{A}, \tau) = \lim_{n\to\infty} \frac{1}{n} H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right)$$

and, if $G \subset \mathcal{M}$ *is an arbitrary set, then the entropy of IF-dynamical system* $(\mathcal{F}, m_\alpha, \tau)$ *is*

$${}_G h_\alpha(\tau) = \sup\{h_\alpha(\mathcal{A}, \tau); \mathcal{A} \text{ is an IF-partition, } \mathcal{A} \subset G\}.$$

Theorem 4.2.

$$h_\alpha(\mathcal{A}, \tau) \leq h_\alpha(\mathcal{C}, \tau) + H_\alpha(\mathcal{A}|\mathcal{C})$$

for any IF-partitions $\mathcal{A}, \mathcal{C}$.

Proof. By Proposition 3.4 we obtain that

$$H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) \le H_\alpha\left[\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right)\vee\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right)\right] =$$

$$= H_\alpha\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right) + H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})|\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right),$$

and

$$H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})|\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right) \le \sum_{i=0}^{n-1} H_\alpha\left(\tau^i(\mathcal{A})|\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right) \le$$

$$\le \sum_{i=0}^{n-1} H_\alpha\left(\tau^i(\mathcal{A})|\tau^i(\mathcal{C})\right) = nH_\alpha(\mathcal{A}|\mathcal{C}).$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}H_\alpha\left(\bigvee_{i=0}^{n-1}\tau^i(\mathcal{A})\right) \le \lim_{n\to\infty}\frac{1}{n}H_\alpha\left(\bigvee_{j=0}^{n-1}\tau^j(\mathcal{C})\right) + H_\alpha(\mathcal{A}|\mathcal{C}).$$

Finally we have

$$h_\alpha(\mathcal{A},\tau) \le h_\alpha(\mathcal{C},\tau) + H_\alpha(\mathcal{A}|\mathcal{C}).$$

□

Theorem 4.3. *Let $\mathcal{C} = \{C_1, ..., C_t\}$ be a measurable partition of Ω being a generator, i.e. $\sigma(\bigcup_{i=0}^{\infty}\tau^i(\mathcal{C})) = \mathcal{S}$. Then for every IF-partition $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ there holds*

$$h_\alpha(\mathcal{A},\tau) \le h_\alpha(\mathcal{C},\tau) + \int_\Omega\left(\sum_{i=1}^{k}(1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i})\right)dP.$$

Proof. See [9]. □

Of course the notion of the IF-entropy has the following defect. If

$$G = \{(\mu, 1-\mu); \mu \in [0,1]\},$$

then ${}_Gh_\alpha(\tau) = \infty$ ([5]). To eleminate this defect Maličký–Riečan modification [6] and also Hudetz modification [5] was used.

5 IF-generators

In the previous section we mention Theorem 4.3 about the set generators $\mathcal{C} = \{C_1, ..., C_t\}$. We shall now prove a theorem about an IF-generator. We shall take as a measure of proximity the operation

$$A \vartriangle B = A \odot B^* + B \odot A^*.$$

Definition 5.1. *An IF-partition $\mathcal{C}$ is called an IF-generator, if to every $\lambda > 0$ and every $A \in F$ there is an element $B \in \bigcup_{i=0}^{\infty}\tau^i(\mathcal{C})$ such that*

$$m(A \vartriangle B) < \lambda.$$

For a set $\mathcal{B} \subset \mathcal{M}$ denote by $s(\mathcal{B}) \supset \mathcal{B}$ the minimal set of elements of $\mathcal{M}$ closed under the complement A^*, Łukasiewicz sum $A \oplus B$ and the maximum $A \vee B$.

Proposition 5.2. *To a given IF-partition $\mathcal{A} = \{A_1, ..., A_n\}$ and every $\delta > 0$ there exists $\lambda > 0$ such that for every $\mathcal{B} = \{B_1, ..., B_n\}$ with $m(A_i \bigtriangleup B_j) < \lambda$ $(i, j = 1, 2, ..., n)$ there is an IF-partition $\{C_1, ..., C_n\} \subset s(\mathcal{B})$ such that $m(A_i \bigtriangleup C_j) < \delta$ $(i, j = 1, 2, ..., n)$.*

Proof. Put $C_1 = B_1$, $C_i = B_i \wedge \left(\sum_{k=1}^{i-1} C_k\right)^*$ and $C_n = \left(\sum_{k=1}^{n-1} C_k\right)^*$. Evidently $\{C_1, ..., C_n\}$ is an IF-partition included in $s(\mathcal{B})$. Let $i = j \in \{1, 2, ..., n-1\}$. If $C_i = B_i$, then $A_i \bigtriangleup C_i = A_i \bigtriangleup B_i$. Conversely, if $C_i = \left(\sum_{k=1}^{i-1} C_k\right)^* \leq B_i$, then

$$C_i \odot A_i^* \leq B_i \odot A_i^* \leq A_i \bigtriangleup B_i,$$

$$A_i \odot C_i^* = A_i \odot \left(\sum_{k=1}^{i-1} C_k\right) = \sum_{k=1}^{i-1} A_i \odot C_k \leq \sum_{k=1}^{i-1} A_k^* \odot C_k \leq \sum_{k=1}^{i-1} A_k \bigtriangleup C_k.$$

So that

$$A_i \bigtriangleup C_i \leq A_i \bigtriangleup B_i + \sum_{k=1}^{i-1} A_k \bigtriangleup C_k,$$

which implies

$$m(A_i \bigtriangleup C_i) \leq m(A_i \bigtriangleup B_i) + \sum_{k=1}^{i-1} m(A_k \bigtriangleup C_k).$$

Now let $i = j = n$. Then $C_n = \left(\sum_{k=1}^{n-1} C_k\right)^*$ and

$$A_n \odot C_n^* = A_n \odot \left(\sum_{k=1}^{n-1} C_k\right) = \sum_{k=1}^{n-1} A_n \odot C_k \leq \sum_{k=1}^{n-1} A_k^* \odot C_k \leq \sum_{k=1}^{n-1} A_k \bigtriangleup C_k,$$

$$C_n \odot A_n^* = C_n \odot \left(\sum_{k=1}^{n-1} A_k\right) = \sum_{k=1}^{n-1} C_n \odot A_k \leq \sum_{k=1}^{n-1} C_k^* \odot A_k \leq \sum_{k=1}^{n-1} C_k \bigtriangleup A_k.$$

Hence

$$A_n \bigtriangleup C_n \leq 2 \sum_{k=1}^{n-1} A_k \bigtriangleup C_k$$

and

$$m(A_n \bigtriangleup C_n) \leq 2 \sum_{k=1}^{n-1} m(A_k \bigtriangleup C_k).$$

Now let $i, j \in \{1, 2, ..., n-1\}; i \neq j$. If $C_j = B_j$, then $A_i \bigtriangleup C_j = A_i \bigtriangleup B_j$. Conversely, if $C_j = \left(\sum_{k=1}^{j-1} C_j\right)^* \leq B_j$, then

$$A_i^* \odot C_j \leq A_i^* \odot B_j \leq A_i \bigtriangleup B_j$$

$$A_i \odot C_j^* = A_i \odot \left(\sum_{k=1}^{j-1} C_k\right) = \sum_{k=1}^{j-1} A_i \odot C_k \leq \sum_{k=1}^{j-1} A_k^* \odot C_k \leq \sum_{k=1}^{j-1} A_k \bigtriangleup C_k.$$

Thus

$$m(A_i \bigtriangleup C_j) \leq m(A_i \bigtriangleup B_j) + \sum_{k=1}^{j-1} m(A_k \bigtriangleup C_k).$$

Now let $i \in \{1, 2, ..., n-1\}$ and $j = n$. Then $C_n = \left(\sum_{k=1}^{n-1} C_k\right)^*$ and

$$A_i \odot C_n^* = A_i \odot \left(\sum_{k=1}^{n-1} C_k\right) = \sum_{k=1}^{n-1} A_i \odot C_k \leq \sum_{k=1}^{n-1} A_k^* \odot C_k \leq \sum_{k=1}^{n-1} A_k \bigtriangleup C_k,$$

$$C_n \odot A_i^* = C_n \odot \left(\sum_{k=1;k\neq i}^{n} A_k\right) = \sum_{k=1;k\neq i}^{n} C_n \odot A_k \leq \sum_{k=1;k\neq i}^{n} C_k^* \odot A_k \leq \sum_{k=1;k\neq i}^{n} C_k \bigtriangleup A_k.$$

So that

$$m(A_i \bigtriangleup C_n) \leq \sum_{k=1}^{n-1} m(A_k \bigtriangleup C_k) + \sum_{k=1;k\neq i}^{n} m(C_k \bigtriangleup A_k).$$

Finally, let $i = n$ and $j \in \{1, 2, ..., n-1\}$. If $C_j = B_j$, then $A_n \bigtriangleup C_j = A_n \bigtriangleup B_j$. Conversely, if $C_j = \left(\sum_{k=1}^{j-1} C_k\right)^* \leq B_j$, then

$$A_n^* \odot C_j \leq A_n^* \odot B_j \leq A_n \bigtriangleup B_j$$

$$A_n \odot C_j^* = A_n \odot \left(\sum_{k=1}^{j-1} C_k\right) = \sum_{k=1}^{j-1} A_n \odot C_k \leq \sum_{k=1}^{j-1} A_k^* \odot C_k \leq \sum_{k=1}^{j-1} A_k \bigtriangleup C_k$$

Hence

$$m(A_n \bigtriangleup C_j) \leq m(A_n \bigtriangleup B_j) + \sum_{k=1}^{j-1} m(A_k \bigtriangleup C_k)$$

Since $m(A_i \bigtriangleup B_j) < \lambda$ $(i, j = 1, 2, ..., n)$, we obtain

$$m(A_i \bigtriangleup C_j) < 2^{\min\{i,j\}-1}\lambda$$

for all $i, j \in \{1, 2, ..., n\}$. Therefore we can put $\lambda = \delta/2^{n-1}$. □

Proposition 5.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $H_\alpha(\mathcal{A}|\mathcal{D}) < \varepsilon$ for any IF-partitions $\mathcal{A} = \{A_1, ..., A_n\}$, $\mathcal{D} = \{D_1, ..., D_n\}$ satisfying the condition*

$$m(A_i \bigtriangleup D_j) < \delta$$

for all $i, j \in \{1, 2, ..., n\}$

Proof. First choose $\delta_0 \in (0, 1)$ such that $\varphi(t) < \varepsilon/n$ for every $t \notin (\delta_0, 1-\delta_0)$ and put

$$\delta = \bigwedge\left\{\frac{\delta_0}{2} m(A_i); m(A_i) > 0\right\}.$$

Then

$$m(A_i) \le m(A_i \triangle D_j) + m(D_j) < \delta + m(D_j) \le \delta_0 \frac{m(A_i)}{2} + m(D_j),$$
$$\frac{m(A_i)}{2} < m(A_i) - \delta_0 \frac{m(A_i)}{2} < m(D_j),$$
$$m(D_j) - m(A_i \odot D_j) \le m(A_i \triangle D_j) \le \delta \le \delta_0 m(D_j).$$

If we consider an i such that $m(D_j) > 0$, then

$$\frac{m(A_i \odot D_j)}{m(D_j)} > 1 - \delta_0,$$

hence

$$\varphi\left(\frac{m(A_i \odot D_j)}{m(D_j)}\right) < \frac{\varepsilon}{n}.$$

Therefore

$$H_\alpha(\mathcal{A}|\mathcal{D}) = \sum_{i=1}^{n}\sum_{j=1}^{n} m(D_j)\varphi\left(\frac{m(A_i \odot D_j)}{m(D_j)}\right) < \sum_{i=1}^{n}\sum_{i=1}^{n} m(D_j)\frac{\varepsilon}{n} = \sum_{j=1}^{n}\frac{\varepsilon}{n} = \varepsilon. \qquad \square$$

Theorem 5.4. *If $\mathcal{C}$ is an IF-generator, then*

$$h_\alpha(\tau) = h_\alpha(\mathcal{C}, \tau).$$

Proof. Let $\mathcal{A} = \{A_1, ..., A_n\}$ be IF-partition, $\mathcal{C}_n = \bigcup_{i=0}^{n} \tau^i(\mathcal{C})$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ according to Proposition 5.3 and $\lambda > 0$ according to Proposition 5.2. Since $\mathcal{C}$ is an IF-generator, there are $B_1, ..., B_n \in \bigcup_{i=0}^{\infty} \tau^i(\mathcal{C})$ such that for all $i, j \in \{1, 2, ..., n\}$

$$m(A_i \triangle B_j) < \lambda.$$

Evidently there exists $k \in \mathbb{N}$ such that $B_1, ..., B_n \in \bigcup_{i=0}^{k} \tau^i(\mathcal{C}) = \mathcal{C}_k$. Put $\mathcal{B} = \{g_1, ..., g_n\}$. By Proposition 5.2 there is a partition $\mathcal{D} = \{D_1, ..., D_n\} \subset s(\mathcal{B}) \subset s(\mathcal{C}_k)$ such that for all $i, j \in \{1, 2, ..., n\}$

$$m(A_i \triangle D_j) < \delta$$

hence

$$H_\alpha(\mathcal{A}|\mathcal{D}) < \varepsilon$$

by Proposition 5.3. By Theorem 4.2

$$h_\alpha(\mathcal{A}, \tau) \le h_\alpha(\mathcal{D}, \tau) + H_\alpha(\mathcal{A}|\mathcal{D}) < h_\alpha(\mathcal{D}, \tau) + \varepsilon.$$

Of course

$$h_\alpha(\mathcal{D}, \tau) \le h_\alpha(\mathcal{C}_k, \tau) = h_\alpha(\mathcal{C}, \tau).$$

Since

$$h_\alpha(\mathcal{A}, \tau) \le h_\alpha(\mathcal{C}, \tau) + \varepsilon$$

for every ε and $\mathcal{C}$ does not depend on ε, we obtain

$$h_\alpha(\mathcal{A}, \tau) \le h_\alpha(\mathcal{C}, \tau)$$

for every IF-partition $\mathcal{A}$. So we have

$$h_\alpha(\tau) = h_\alpha(\mathcal{C}, \tau). \qquad \square$$

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