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On $\mathcal{I}$-lacunary summability methods of order α in intuitionistic fuzzy 2-normed spaces

Ekrem Savaş

Department of Mathematics, Istanbul Ticaret University
Sütlüce-Istanbul, Turkey
e-mail: esavas@ticaret.edu.tr

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Abstract: In this paper, we introduce and study the notion $\mathcal{I}$-statistical convergence of order α, and $\mathcal{I}$-lacunary statistical convergence of order α with respect to the intuitionistic fuzzy 2-normed space, investigate their relationship and also we have proved some inclusion theorems.
Keywords: Ideal, Filter, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence, Statistical convergence of order α.
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1 Introduction

Zadeh [40] introduced the concept of fuzzy sets and fuzzy set operations. Subsequently several authors have discussed various aspects of its theory and applications such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming etc. In recent years, there has been an increasing interest in various mathematical aspects of operations defined on fuzzy sets. Based on these, sequences of fuzzy numbers have been introduced by several authors and they have obtained many important properties.

The class of all p-summable convergent sequences of fuzzy numbers is introduced by Nanda [18]. Later on statistical convergence of sequences of fuzzy numbers are introduced by Nuray and Savas (see, [19]). Recently bounded variation for fuzzy numbers is studied by Tripathy et al. in [38, 39]. As the set of all real numbers can be embedded in the set of all fuzzy numbers, many results in reals can be considered as a special case of those fuzzy numbers. However, since the set of all fuzzy numbers is partially ordered and does not carry a group structure, most of the facts known for the sequences of real numbers may not valid in fuzzy setting. Therefore, this theory is not a trivial extension of what has been known in real case.

The theory of intuitionistic fuzzy sets was introduced by Atanassov [1]; it has been extensively used in decision making problems [2]. The concept of an intuitionistic fuzzy metric space was introduced by Park [20]. Furthermore, Saadati and Park [21] gave the notion of an intuitionistic fuzzy normed space. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in [11, 32, 33, 37].

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [7] (see also [35]) as follows. Let K be a subset of $\mathbb{N}$. Then the asymptotic density of K is denoted by $\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L we write $st\text{-}\lim x_k = L$. It is doubtless that the study of statistical convergence and its various generalizations has become an active research area since late 90's of the last century. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [8] and Šalát [22].

In [12], P. Kostyrko et al. introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [5, 6, 24, 26, 27, 28, 29, 31].

In another direction, a new type of convergence, called lacunary statistical convergence, was introduced in [9]. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or S_θ-convergent to L) if, for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [9], the relation between lacunary statistical convergence and statistical convergence was established, among other things. Recently, Mohiuddine and Aiyub [14] introduced the concept lacunary statistical convergence in random 2-normed space. In [15], Mursaleen and Mohiuddine extended the idea of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. Also lacunary statistically convergent double sequences in probabilistic normed space was studied by Mohiuddine and Savaş in [13]. In [32], generalized statistical convergence in intuitionistic fuzzy 2-normed space was studied by Savas. Mursaleen, et al. studied the ideal convergence of double sequences in intuitionistic fuzzy normed spaces (see, [17]). More results on this convergence can be found in [25, 30].

Recently in [23] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence with respect to the intuitionistic fuzzy norm (μ, v) which naturally extend the notions of the above mentioned convergence.

On the other hand, in [3, 4] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by replacing n by n^{α} in the denominator in the definition of statistical convergence. One can also see [5, 27, 33] for some related works

In this paper, the notions of ideal statistical convergence of order α and ideal lacunary statistical convergence of order α, where $0 < \alpha < 1$ are introduced in an intuitionistic fuzzy normed linear space and some important results are obtained. Finally, we try to establish the relation between these two summability notions.

Throughout the paper, $\mathbb{N}$ will denote the set of all natural numbers. First we need some basic definitions used in the paper.

Definition 1. *A triangular norm (t-norm) is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in [0,1]$.*

Definition 2 ([36])**.** *A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*

(i) *$\Diamond$ is associate and commutative,*

(ii) *$\Diamond$ is continuous,*

(iii) *$a \Diamond 0 = a$ for all $a \in [0,1]$,*

(iv) *$a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.*

For example, we can give $a * b = ab$, $a * b = \min\{a, b\}$, $a \Diamond b = \min\{a + b, 1\}$ and $a \Diamond b = \max\{a, b\}$ for all $a, b \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [21] has recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 3 ([21])**.** *The 5-tuple $(X, \mu, v, *, \Diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and μ, v are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$, and $s, t > 0$:*

(a) $\mu(x,t) + v(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$,

(d) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, t + s)$,

(f) $\mu(x,.) : (0, \infty) \to [0,1]$ *is continuous,*

(g) $\lim\limits_{t \to \infty} \mu(x,t) = 1$ *and* $\lim\limits_{t \to 0} \mu(x,t) = 0$,

(h) $v(x,t) < 1$,

(i) $v(x,t) = 0$ *if and only if* $x = 0$,

(j) $v(\alpha x,t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(k) $v(x,t)\lozenge v(y,s) \geq v(x+y,t+s)$,

(l) $v(x,.) : (0,\infty) \to [0,1]$ *is continuous,*

(m) $\lim_{t\to\infty} v(x,t) = 0$ *and* $\lim_{t\to 0} v(x,t) = 1$.

In this case (μ, v) is called an intuitionistic fuzzy norm.

In [10], Gähler introduced the following concept of 2-normed space.

Definition 4. *Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $||.|| : X \times X \to \mathbb{R}$ which satisfies,*

(a) $||x,y|| = 0$ *if and only if x and y are linearly dependent;*

(b) $||x,y|| = ||y,x||$*;*

(c) $||\alpha x,y|| = |\alpha|||x,y||$;

(d) $||x,y+z|| \leq ||x,y|| + ||x,z||$.

The pair $(X, ||.,.||)$ is then called a 2-normed space.

A trivial example of an 2-normed space is $X = \mathbf{R}^2$, equipped with the Euclidean 2-norm $||x_1,x_2||_E =$ the volume of the parallellogram spanned by the vectors x_1, x_2 which may be given expicitly by the formula

$$||x_1,x_2||_E = |det(x_{ij})| = \text{abs}\left(\det(< x_i, x_j >)\right)$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbf{R}^2$ for each $i = 1, 2$.

Mursaleen and Lohani [16] used the idea of 2-normed space to define the intuitionistic fuzzy 2-normed space.

Definition 5. *The five-tuple $(X, \mu, v, *, \lozenge)$ is said to be an intuitionistic fuzzy 2-norm space (for short, IF2NS) if X is a vector space, $*$ is continuous t-norm, $\lozenge$ is continuous t-conorm, and μ, v are fuzzy sets on $X \times X \to (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$.*

(a) $\mu(x,y;t) + v(x,y;t) \leq 1$,

(b) $\mu(x,y;t) > 0$,

(c) $\mu(x,y;t) = 1$ *if and only if x and y are linearly dependent,*

(d) $\mu(\alpha x,y;t) = \mu(x,y;\frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,

(e) $\mu(x, y; t) * \mu(x, z; s) \leq \mu(x, y + z; t + s)$,

(f) $\mu(x, y; .) : (0, \infty) \to [0, 1]$ *is continuous,*

(g) $\lim_{t\to\infty} \mu(x, y; t) = 1$ *and* $\lim_{t\to 0} \mu(x, y : t) = 0$,

(h) $\mu(x, y; t) = \mu(y, x; t)$

(i) $v(x, y; t) < 1$,,

(j) $v(x, y; t) = 0$ *if and only if* x *and* y *are linearly dependent,*

(k) $v(\alpha x, y; t) = v(x, y; \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,

(l) $\mu(x, y; t)\Diamond\mu(x, z; s) \geq v(x, y + z; t + s)$,

(m) $v(x, y; .) : (0, \infty) \to [0, 1]$ *is continuous,*

(n) $\lim_{t\to\infty} v(x, y; t) = 0$ *and* $\lim_{t\to 0} v(x, y; t) = 1$.

(o) $v(x, y; t) = v(y, x; t)$

In this case (μ, v) is called an intuitionistic fuzzy 2-norm on X, and we denote it by $(\mu, v)_2$.

Example 1. *Let* $(X, ||.||)$ *be a* 2*-normed space, and let* $a * b = ab$ *and* $a\Diamond b = \min\{a + b, 1\}$ *for all* $a, b \in [0, 1]$ *. For all* $x \in X$ *and every* $t > 0$*, consider*

$\mu(x, z; t) := \frac{t}{t+||x,z||}$ *and* $v(x, z; t) := \frac{||x,z||}{t+||x,z||}$

Then $(X, \mu, v, *, \Diamond)$ *is an intuitionistic fuzzy* 2*-normed space.*

We also recall that the concept of convergence in an intuitionistic fuzzy 2-normed space is studied in [16].

Definition 6 ([16])**.** *Let* $(X, \mu, v, *, \Diamond)$ *be an intuitionistic fuzzy* 2*-normed space. Then, a sequence* $x = (x_k)$ *is said to be convergent to* $L \in X$ *with respect to* $(\mu, v)_2$ *if, for every* $\varepsilon > 0$ *and* $t > 0$*, there exists* $k_0 \in \mathbb{N}$ *such that* $\mu(x_k - L, z; t) > 1 - \varepsilon$ *and* $v(x_k - L, z; t) < \varepsilon$ *for all* $k \geq k_0$ *and for all* $z \in X$*. It is denoted by* $(\mu, v)_2 - \lim x = L$ *or* $x_k \overset{(\mu,v)_2}{\to} L$ *as* $k \to \infty$*.*

2 $\mathcal{I}$-Statistical and $\mathcal{I}$-Lacunary statistical convergence of order α on IF2NS

In this section, we deal with the ideal statistical convergence of order α and ideal lacunary statistical convergence of order α on the intuitionistic fuzzy norm spaces. Before proceeding further, we should recall some notations on the ideal.

Definition 7. *A family* $\mathcal{I} \subset 2^{\mathbb{N}}$ *is said to be an ideal of* $\mathbb{N}$ *if the following conditions hold:*

(a) $A, B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$,

(b) $A \in \mathcal{I},\ B \subset A$ *implies* $B \in \mathcal{I}$.

Definition 8. *A non-empty family* $F \subset 2^{\mathbb{N}}$ *is said to be an filter of* $\mathbb{N}$ *if the following conditions hold:*

(a) $\phi \notin F$,

(b) $A, B \in F$ *implies* $A \cap B \in F$,

(c) $A \in F,\ A \subset B$ *implies* $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists\, A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 9. *A proper ideal* $\mathcal{I}$ *is said to be admissible if* $\{n\} \in \mathcal{I}$ *for each* $n \in \mathbb{N}$.

Throughout, $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 10 ([12])**.** *Let* $\mathcal{I} \subset 2^{\mathbb{N}}$ *be a proper admissible ideal in* $\mathbb{N}$. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of elements of* $\mathbb{R}$ *is said to be* $\mathcal{I}$*-convergent to* $L \in \mathbb{R}$ *if, for each* $\varepsilon > 0$, *the set* $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$.

Definition 11 ([5])**.** *A sequence* $x = \{x_k\}_{k\in\mathbb{N}}$ *is said to be* $\mathcal{I}$*-statistically convergent to* L *or* $S(\mathcal{I})$*-convergent to* L *if, for each* $\varepsilon > 0$ *and* $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left| \{k \leq n : \|x_k - L\| \geq \varepsilon\} \right| \geq \delta \right\} \in \mathcal{I}$$

or equivalently if for each $\varepsilon > 0$

$$\delta_{\mathcal{I}}(A(\varepsilon)) = \mathcal{I}\text{-}\lim \delta_n(A(\varepsilon)) = 0,$$

where $A(\varepsilon) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ *and* $\delta_n(A(\varepsilon)) = \frac{|A(\varepsilon)|}{n}$.

In this case we write $x_k \to L\,(S(\mathcal{I}))$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted simply by $S(\mathcal{I})$. Let $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbf{N} : A \text{ is a finite subset}\}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}$-statistically convergent is the statistical convergence.

We now ready to obtain our main definitions and results.

Definition 12. *Let* $(X, \mu, v, *, \Diamond)$ *be an intuitionistic fuzzy* 2*-normed space. Then, a sequence* $x = (x_k)$ *is said to be* $\mathcal{I}$*-statistically convergent of order* α *to* $L \in X$ *or* $S(\mathcal{I})^{\alpha}$*-convergent to* L, *where* $0 < \alpha \leq 1$, *with respect to* $(\mu, v)_2$ *if, for each* $\varepsilon > 0$, $t > 0$ *and* $\delta > 0$, *and for non zero* $z \in X$ *such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \varepsilon\}\right| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $x_k \overset{(\mu,v)_2}{\to} L\left(S^{(\mu,v)_2}(\mathcal{I})^{\alpha}\right)$. *The class of all* $\mathcal{I}$*-statistically convergent of order* α *sequences will be denoted by simply* $S^{(\mu,v)_2}(\mathcal{I})^{\alpha}$.

Remark 1. *For $\mathcal{I} = \mathcal{I}_f = \{A \subseteq \mathbf{N} : A$ is a finite subset$\}$. $S^{(\mu,v)_2}(\mathcal{I})^{\alpha}$-convergence coincides with statistically convergence of order α with respect to $(\mu, v)_2$. For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$ it coincides with $\mathcal{I}$-statistically convergence with respect to $(\mu, v)_2$ [34]. When $I = I_{fin}$ and $\alpha = 1$ it becomes only statistically convergence with respect to $(\mu, v)_2$ (see, [32]).*

Definition 13. *Let $(X, \mu, v, *, \Diamond)$ be an intuitionistic fuzzy 2-normed space and θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $\mathcal{I}$-lacunary statistically convergent of order α to $L \in X$ or $S_\theta(\mathcal{I})^\alpha$-convergent to L, where $0 < \alpha \leq 1$, with respect to$(\mu, v)_2$ if, for any $\varepsilon > 0$, $t > 0$ and $\delta > 0$, and for non zero $z \in X$ such that*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\{k \in \mathcal{I}_r : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \overset{(\mu,v)_2}{\to} L \left(S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$. The class of all $\mathcal{I}$-lacunary statistically convergent sequences will be denoted by $S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha$.

Remark 2. *For $\alpha = 1$ the definition coincides with $\mathcal{I}$-lacunary statistical convergence with respect to $(\mu, v)_2$, [34]. Further it must be noted in this context that lacunary statistical convergence of order α with respect to $(\mu, v)_2$ has not been studied till now. Obviously lacunary statistical convergence of order α with respect to $(\mu, v)_2$ is a special case of $\mathcal{I}$-lacunary statistical convergence of order α with respect to $(\mu, v)_2$ when we take $\mathcal{I} = \mathcal{I}_f$. So properties of lacunary statistical convergence of order α with respect to $(\mu, v)_2$ can be easily obtained from our results with obvious modifications.*

Theorem 1. *Let $(X, \mu, v, *, \Diamond)$ be an intuitionistic fuzzy 2-normed space and $0 < \alpha \leq \beta \leq 1$. Then $S^{(\mu,v)_2}(\mathcal{I})^\alpha \subset S^{(\mu,v)_2}(\mathcal{I})^\beta$.*

Proof. Let $(X, \mu, v, *, \Diamond)$ be an intuitionistic fuzzy 2-normed space and $0 < \alpha \leq \beta \leq 1$. Then

$$\begin{aligned}&\frac{|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}|}{n^\beta} \\ \leq\; &\frac{|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}|}{n^\alpha}\end{aligned}$$

and so for any $\delta > 0$, and for non zero $z \in X$

$$\begin{aligned}&\{n \in \mathbb{N} : \frac{|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}|}{n^\beta} \geq \delta\} \subset \\ &\{n \in \mathbb{N} : \frac{|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}|}{n^\alpha} \geq \delta\}.\end{aligned}$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S^{(\mu,v)_2}(\mathcal{I})^\alpha \subset S^{(\mu,v)_2}(\mathcal{I})^\beta$. □

Corollary 1. *If a sequence is $\mathcal{I}$-statistically convergent of order α to L for some $0 < \alpha \leq 1$ with respect to $(\mu, v)_2$ then it is $\mathcal{I}$-statistically convergent to L with respect to $(\mu, v)_2$ i.e. $S^{(\mu,v)_2}(\mathcal{I})^\alpha \subset S^{(\mu,v)_2}(\mathcal{I})$.*

Similarly we can show that

Theorem 2. *Let* $0 < \alpha \leq \beta \leq 1$. *Then,*

(i) $S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha \subset S_\theta^{(\mu,v)_2}(\mathcal{I})^\beta$.

(ii) *In particular* $S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha \subset S_\theta^{(\mu,v)_2}(\mathcal{I})$.

It can be checked that $S^{(\mu,v)_2}(\mathcal{I})^\alpha$ and $S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha$ are both linear subspaces the space of $(X, \mu, v, *, \Diamond)$. We now prove the following result which gives a topological characterization of these spaces. As the line of proofs for both are similar we give the detailed proof for the class $S^{(\mu,v)_2}(\mathcal{I})^\alpha$ only.

Theorem 3. $S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha \cap l_\infty^{(\mu,v)_2}$ *is a closed subset of* $l_\infty^{(\mu,v)_2}$, *where* $l_\infty^{(\mu,v)_2}$ *stands for the space of all bounded sequences of intuitionistic fuzzy 2-normed space.*

Proof. Suppose that $\{x^n\}_{n\in\mathbb{N}} \subseteq S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha \cap l_\infty^{(\mu,v)_2}$ is a convergent sequence and that it converges to $x \in l_\infty^{(\mu,v)_2}$. We need to prove that $x \in S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha \cap l_\infty^{(\mu,v)_2}$. Assume that $x^n \to L_n\left(S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$, $\forall n \in \mathbb{N}$. Take a positive strictly decreasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ where $\varepsilon_n = \frac{\varepsilon}{2^n}$ for a given $\varepsilon > 0$. Clearly $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converges to 0. Choose a positive integer n such that $\|x - x^n\|_\infty = \sup_n \{v(x - x^n, t)\} < \frac{\varepsilon_n}{4}$. Let $0 < \delta < 1$. Then

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \left\{ \begin{array}{c} k \in I_r : \mu(x_k^n - L_n, z; t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or} \\ v(x_k^n - L_n, z; t) \geq \frac{\varepsilon_n}{4} \end{array} \right\} \right| < \frac{\delta}{3} \right\} \in F(\mathcal{I})$$

and

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \left\{ \begin{array}{c} k \in I_r : \mu\left(x_k^{n+1} - L_{n+1}, z; t\right) \leq 1 - \frac{\varepsilon_{n+1}}{4} \text{ or} \\ v\left(x_k^{n+1} - L_{n+1}, z; t\right) \geq \frac{\varepsilon_{n+1}}{4} \end{array} \right\} \right| < \frac{\delta}{3} \right\} \in F(\mathcal{I}).$$

Since $A \cap B \in F(\mathcal{I})$ and $\emptyset \notin F(\mathcal{I})$, we can choose $r \in A \cap B$. Then

$$\frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \mu(x_k^n - L_n, z; t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or } v(x_k^n - L_n, z; t) \geq \frac{\varepsilon_n}{4} \right\} \right| < \frac{\delta}{3}$$

and

$$\frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \mu\left(x_k^{n+1} - L_{n+1}, z; t\right) \leq 1 - \frac{\varepsilon_{n+1}}{4} \text{ or } v\left(x_k^{n+1} - L_{n+1}, z; t\right) \geq \frac{\varepsilon_{n+1}}{4} \right\} \right| < \frac{\delta}{3}$$

and so

$$\frac{1}{h_r^\alpha} \left| \left\{ \begin{array}{c} k \in I_r : \mu(x_k^n - L_n, z; t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or } v(x_k^n - L_n, z; t) \geq \frac{\varepsilon_n}{4} \vee \\ \mu\left(x_k^{n+1} - L_{n+1}, z; t\right) \leq 1 - \frac{\varepsilon_{n+1}}{4} \text{ or } v\left(x_k^{n+1} - L_{n+1}, z; t\right) \geq \frac{\varepsilon_{n+1}}{4} \end{array} \right\} \right| < \delta < 1.$$

Hence, there exists a $k \in I_r$ for which $\mu(x_k^n - L_n, z; t) > 1 - \frac{\varepsilon_n}{4}$ or $v(x_k^n - L_n, z; t) < \frac{\varepsilon_n}{4}$ and $\mu\left(x_k^{n+1} - L_{n+1}, z; t\right) > 1 - \frac{\varepsilon_{n+1}}{4}$ or $v\left(x_k^{n+1} - L_{n+1}, z; t\right) < \frac{\varepsilon_{n+1}}{4}$.

Then, we can write

$$\begin{aligned}
v\left(L_n - L_{n+1}, z; t\right) &\leq v\left(L_n - x_k^n, z; \frac{t}{3}\right) \Diamond v\left(x_k^n - x_k^{n+1}, z; \frac{t}{3}\right) \Diamond v\left(x_k^{n+1} - L_{n+1}, z; \frac{t}{3}\right) \\
&\leq v\left(x_k^n - L_n, z; \frac{t}{3}\right) \Diamond v\left(x_k^{n+1} - L_{n+1}, z; \frac{t}{3}\right) \Diamond \sup_n \left\{v\left(x - x^n, z; t\right)\right\} \\
&\Diamond \sup_n \left\{v\left(x - x^{n+1}, z; t\right)\right\} \\
&\leq \frac{\varepsilon_n}{4} \Diamond \frac{\varepsilon_{n+1}}{4} \Diamond \frac{\varepsilon_n}{4} \Diamond \frac{\varepsilon_{n+1}}{4} \leq \varepsilon_n.
\end{aligned}$$

This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_n \to L$, as $n \to \infty$. We need to prove that $x \to L\left(S_\theta^{(\mu,v)}(\mathcal{I})^\alpha\right)$. For any $\varepsilon > 0$ and $t > 0$, choose $n \in \mathbb{N}$ such that $\varepsilon_n < \frac{\varepsilon}{4}$, $\|x - x^n\|_\infty = \sup_n \{v(x - x^n, z; t)\} < \frac{\varepsilon}{4}$, $\mu(L_n - L, z; t) > 1 - \frac{\varepsilon}{4}$ or $v(L_n - L, z; t) < \frac{\varepsilon}{4}$. Then

$$\begin{aligned}
&\frac{1}{h_r^\alpha} \left|\left\{k \in I_r : v\left(x_k - L, z; t\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{h_r^\alpha} \left|\left\{\begin{array}{c} k \in I_r : v\left(x_k^n - L_n, z; t\right) \Diamond \sup_n \left\{v\left(x_k - x_k^n, z; t\right)\right\} \\ \Diamond v\left(L_n - L, z; t\right) \geq \varepsilon \end{array}\right\}\right| \\
&\leq \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : v\left(x_k^n - L_n, z; t\right) \Diamond \frac{\varepsilon}{4} \Diamond \frac{\varepsilon}{4} \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : v\left(x_k^n - L_n, z; t\right) \geq \frac{\varepsilon}{2}\right\}\right|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\right\}\right| < \delta\right\} \\
&\supseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : \mu\left(x_k^n - L_n, z; t\right) \leq 1 - \frac{\varepsilon}{2} \text{ or } v\left(x_k^n - L_n, z; t\right) \geq \frac{\varepsilon}{2}\right\}\right| < \delta\right\} \in F(\mathcal{I}).
\end{aligned}$$

So

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\right\}\right| < \delta\right\} \in F(\mathcal{I}),$$

and so

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

This gives that $x \to L\left(S_\theta^{(\mu,v)}(\mathcal{I})^\alpha\right)$, and this completes the proof of the theorem. □

Theorem 4. *$S^{(\mu,v)_2}(\mathcal{I})^\alpha \cap l_\infty^{(\mu,v)_2}$ is a closed subset of $l_\infty^{(\mu,v)_2}$.*

Definition 14. *Let θ be a lacunary sequence. Then $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $N_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha$-convergent to $L \in X$ with respect to $(\mu, v)_2$ if, for any $\varepsilon > 0$, $t > 0$ and $\delta > 0$, and for non zero $z \in X$ such that*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\sum_{k\in I_r}\mu\left(x_k-L,z;t\right)\le 1-\varepsilon \text{ or } \frac{1}{h_r^\alpha}\sum_{k\in I_r}v\left(x_k-L,z;t\right)\ge\varepsilon\right\}\in\mathcal{I}.$$

This convergence is denoted by $x_k\to L\left(N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha\right)$*, and the class of such sequences will be denoted simply by* $N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha$.

Theorem 5. *Let* θ *be a lacunary sequence. Then*

(a) $x_k\to L\left(N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha\right)\Rightarrow x_k\to L\left(S_\theta^{(\mu,v)}\left(\mathcal{I}\right)^\alpha\right)$, *and*

(b) $N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha$ *is a proper subset of* $S_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha$.

Proof. (a) If $\varepsilon>0$ and $x_k\to L\left(N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha\right)$, we can write

$$\begin{aligned}&\sum_{k\in I_r}\left(\mu\left(x_k-L,z;t\right)\text{ or } v\left(x_k-L,z;t\right)\right)\\ &\ge\sum_{\substack{k\in I_r,\mu(x_k-L,z;t)\le 1-\varepsilon\\ \text{or } v(x_k-L,z;t)\ge\varepsilon}}\left(\mu\left(x_k-L,z;t\right)\text{ or } v\left(x_k-L,z;t\right)\right)\\ &\ge\varepsilon\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\le 1-\varepsilon\text{ or } v\left(x_k-L,z;t\right)\ge\varepsilon\right\}\right|\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{\varepsilon h_r^\alpha}\sum_{k\in I_r}\left(\mu\left(x_k-L,z;t\right)\text{ or } v\left(x_k-L,z;t\right)\right)\\ &\ge\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\le 1-\varepsilon\text{ or } v\left(x_k-L,z;t\right)\ge\varepsilon\right\}\right|.\end{aligned}$$

Then, for any $\delta>0$ and $t>0$, and for non zero $z\in X$

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\le 1-\varepsilon\text{ or } v\left(x_k-L,z;t\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\ &\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mu\left(x_k-L,z;t\right)\le\left(1-\varepsilon\right)\delta\text{ or }\frac{1}{h_r}\sum_{k\in I_r}v\left(x_k-L,z;t\right)\ge\varepsilon.\delta\right\}\in\mathcal{I}.\end{aligned}$$

This proves the result.

(b) In order to establish that the inclusion $N_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha\subseteq S_\theta^{(\mu,v)_2}\left(\mathcal{I}\right)^\alpha$ is proper, let θ be given, and define x_k to be $1,2,...,\left[\sqrt{h_r}\right]$ for the first $\left[\sqrt{h_r}\right]$ integers in I_r and $x_k=0$ otherwise, for all $r=1,2,...$. Then, for any $\varepsilon>0$ and $t>0$, and for non zero $z\in X$

$$\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-0,z;t\right)\le 1-\varepsilon\text{ or } v\left(x_k-0,z;t\right)\ge\varepsilon\right\}\right|\le\frac{\left[\sqrt{h_r}\right]}{h_r},$$

and for any $\delta>0$, and for non zero $z\in X$ we get

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-0,z;t\right)\le 1-\varepsilon\text{ or } v\left(x_k-0,z;t\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\ &\subseteq\left\{r\in\mathbb{N}:\frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha}\ge\delta\right\}.\end{aligned}$$

Since the set on the right-hand side is a finite set and so belongs to $\mathcal{I}$, it follows that $x_k \to 0\left(S_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$.

On the other hand,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left(\mu\left(x_k-0,z;t\right) \text{ or } v\left(x_k-0,z;t\right)\right)=\frac{1}{h_r}.\frac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r^\alpha}\right]+1\right)}{2}.$$

Then

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\sum_{k\in I_r}\mu\left(x_k-0,z;t\right)\leq 1-\frac{1}{4} \text{ or } \frac{1}{h_r^\alpha}\sum_{k\in I_r}v\left(x_k-0,z;t\right)\geq\frac{1}{4}\right\}$$
$$=\left\{r\in\mathbb{N}:\frac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{h_r^\alpha}\geq\frac{1}{2}\right\}=\{m,m+1,m+2,...\}$$

for some $m\in\mathbb{N}$ which belongs to $F(\mathcal{I})$, since $\mathcal{I}$ is admissible. So $x_k \nrightarrow 0\left(N_\theta^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$. □

Remark 3. *In Theorem 1, [34] it was further proved that* (ii) $x\in l_\infty^{(\mu,v)_2}$ *and* $x_k\to L(S_\theta(\mathcal{I}))\Rightarrow x_k\to L(N_\theta(\mathcal{I})$,

(iii)
$$S_\theta^{(\mu,v)_2}(\mathcal{I})\cap l_\infty^{(\mu,v)_2}=N_\theta^{(\mu,v)_2}(\mathcal{I})\cap l_\infty^{(\mu,v)_2}.$$

However, whether these results remain true for $0<\alpha<1$ *is not clear and we leave them as open problems.*

We will now investigate the relationship between $\mathcal{I}$-statistical and $\mathcal{I}$-lacunary statistical convergence of order α.

Theorem 6. *Let* $(X,\mu,v,*,\Diamond)$ *be an intuitionistic fuzzy* 2*-normed space. For any lacunary sequence* θ*,* $\mathcal{I}$*-statistical convergence of order* α *with respect to* $(\mu,v)_2$ *implies* $\mathcal{I}$*-lacunary statistical convergence of order* α *with respect to* $(\mu,v)_2$ *if* $\liminf_r q_r^\alpha>1$.

Proof. Suppose first that $\liminf_r q_r^\alpha>1$. Then there exists $\sigma>0$ such that $q_r^\alpha\geq 1+\sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha}\geq\frac{\sigma}{1+\sigma}.$$

Since $x_k\to L\left(S^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$, for every $\varepsilon>0$, $t>0$, and for sufficiently large r, we have

$$\frac{1}{k_r^\alpha}\left|\left\{k\leq k_r:\mu\left(x_k-L,z;t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,z;t\right)\geq\varepsilon\right\}\right|$$
$$\geq\frac{1}{k_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,z;t\right)\geq\varepsilon\right\}\right|$$
$$\geq\frac{\sigma}{1+\sigma}.\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,z;t\right)\geq\varepsilon\right\}\right|.$$

Then for any $\delta>0$, and for non zero $z\in X$ we get

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\mu\left(x_k-L,z;t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{r\in\mathbb{N}:\frac{1}{k_r^\alpha}\left|\left\{k\leq k_r:\mu\left(x_k-L,z;t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,z;t\right)\geq\varepsilon\right\}\right|\geq\frac{\delta\alpha}{(1+\alpha)}\right\}\in\mathcal{I}.$$

□

Remark 4. *The converse of this result is true for $\alpha = 1$ (see Theorem 3 [34]). However for $\alpha < 1$ it is not clear and we leave it as an open problem.*

We now present two theorems which specify the sufficient conditions for the converse relation of Theorem 7 to be true. In this context it should be mentioned that it was left as an open problem for the case $\alpha = 1$ (Problem 1 [34]). For the next two results we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(I)$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(I)$.

Theorem 7. *Let $(X, \mu, v, *, \Diamond)$ be an intuitionistic fuzzy 2-normed space. For a lacunary sequence θ satisfying the above condition, $\mathcal{I}$-lacunary statistical convergence with respect to $(\mu, v)_2$ implies $\mathcal{I}$-statistical convergence with respect to $(\mu, v)_2$ if $\limsup\limits_{r} q_r < \infty$.*

Proof. If $\limsup_r q_r < \infty$ then without any loss of generality we can assume that there exists a $0 < B < \infty$ such that $q_r < B$ for all $r \geq 1$. Suppose that $x_k \to L\left(S^{(\mu,v)_2}(\mathcal{I})^{\alpha}\right)$ and for $\epsilon, \delta, \delta_1 > 0$, and for non zero $z \in X$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \epsilon\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \epsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{h_j}|\{k \in I_j : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
&\frac{1}{n}\left|\{k \leq n : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| \\
&\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| \\
&= \frac{1}{k_{r-1}}\left|\{k \in I_1 : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| + \ldots \\
&+ \frac{1}{k_{r-1}}\left|\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}\left|\{k \in I_1 : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| \\
&+ \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}\left|\{k \in I_2 : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, t\right) \geq \varepsilon\}\right| + \cdots + \\
&+ \frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}\left|\{k \in I_r : \mu\left(x_k - L, z; t\right) \leq 1 - \varepsilon \text{ or } v\left(x_k - L, z; t\right) \geq \varepsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r \\
&\leq \sup_{j \in C} A_j . \frac{k_r}{k_{r-1}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. $\square$

Theorem 8. *Let $(X, \mu, v, *, \Diamond)$ be an intuitionistic fuzzy 2-normed space. For a lacunary sequence θ satisfying the above condition, $\mathcal{I}$-lacunary statistical convergence of order α with respect to $(\mu, v)_2$ implies $\mathcal{I}$-statistical convergence of order α, $0 < \alpha < 1$, with respect to $(\mu, v)_2$ if* $\sup\limits_r \sum\limits_{i=0}^{r-1} \frac{h_{i+1}^\alpha}{(k_{r-1})^\alpha} = B(say) < \infty$.

Proof. $x_k \to L\left(S^{(\mu,v)_2}(\mathcal{I})^\alpha\right)$ and for $\epsilon, \delta, \delta_1 > 0$, and for non zero $z \in X$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{{h_r}^\alpha} \left|\{k \in I_r : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{n^\alpha} \left|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{{h_j}^\alpha} \left|\{k \in I_j : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
&\frac{1}{n^\alpha} \left|\{k \leq n : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| \\
&\leq \frac{1}{k_{r-1}^\alpha} \left|\{k \leq k_r : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| \\
&= \frac{1}{k_{r-1}^\alpha} \left|\{k \in I_1 : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| + \dots \\
&+ \frac{1}{k_{r-1}^\alpha} \left|\{k \in I_r : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| \\
&= \frac{k_1^\alpha}{k_{r-1}^\alpha} \frac{1}{h_1^\alpha} \left|\{k \in I_1 : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| \\
&+ \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} \frac{1}{h_2^\alpha} \left|\{k \in I_2 : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| + \dots \\
&+ \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} \frac{1}{h_r^\alpha} \left|\{k \in I_r : \mu(x_k - L, z; t) \leq 1 - \varepsilon \text{ or } v(x_k - L, z; t) \geq \epsilon\}\right| \\
&= \frac{k_1^\alpha}{k_{r-1}^\alpha} A_1 + \frac{(k_2 - k_1)^\alpha}{k_{r-1}^\alpha} A_2 + \dots + \frac{(k_r - k_{r-1})^\alpha}{k_{r-1}^\alpha} A_r \\
&\leq \sup_{j \in C} A_j . \sup_r \sum_{i=0}^{r-1} \frac{(k_{i+1} - k_i)^\alpha}{k_{r-1}^\alpha} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. $\square$

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