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Some algebraic properties of the matrix representation of intuitionistic fuzzy modal operators

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Abstract: Intuitionistic fuzzy sets were introduced by Atanassov in 1983 [1]. He defined Intuitionistic Fuzzy Modal Operators on IFSs. Intuitionistic fuzzy modal operators were separated in three types called one-two-uni type modal operators on Intuitionistic Fuzzy Sets with the help of some authors' studies [3, 5]. A study named Matrix Representation of the Second Type of Intuitionistic Fuzzy Modal Operators introduced in [4]. In this study, our purpose is to examine some new algebraic properties of matrix representation sets of Intuitionistic Fuzzy Modal Operators which is a monoid with the matrix product and give the concept of generator in this algebraic structure.
Keywords: Intuitionisic fuzzy sets, Intuitionistic fuzzy modal operators, Matrix representation of IFMOs on IFSs, Algebraic properties of matrix representation of IFMOs on IFSs.
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1 Introduction

Fuzzy sets introduced by Zadeh [7] is an extension of crisp sets by expanding the truth value set to real unit interval $[0, 1]$. Let X be a fixed set. Function $\mu : X \to [0, 1]$ is a fuzzy set over X. FS(X) represents the class of fuzzy sets over X. For $x \in X$, $\mu(x)$ denotes the membership degree of x and $1 - \mu(x)$ denotes the non-membership degree of x. Intuitionistic fuzzy sets (IFSs) which are an extension of fuzzy sets were introduced by Atanassov in 1983 [1]. The modal operators are important tools for IFSs. The notion of Intuitionistic fuzzy operators was discussed in [2]. Many authors [3, 5] defined new Intuitionistic fuzzy operators and some of their properties were studied. Matrix representation of Intuitionistic fuzzy modal operators was introduced in [4]. In this study, we use the matrix representation of Intuitionistic fuzzy modal operators which is the generalization of the Intuitionistic fuzzy modal operators which are used. In all the studies that have been published until now, none of the Intuitionistic fuzzy modal operators has been determined yet. Another essential point is that the matrix representation of Intuitionistic fuzzy modal operators will be the generalization of Intuitionistic fuzzy modal operators, which will be introduced in the future. In this study, our purpose is to examine those algebraic properties, which have not been investigated for the intuitionistic fuzzy modal operators.

Definition 1 [1]. *Let* $L = [0, 1]$ *then*

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with

$$(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow \text{“} x_1 \leq y_1 \;\; and \;\; x_2 \geq y_2 \text{”}$$

For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows:

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2)),$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2)).$$

Definition 2 [2]. *An intuitionistic fuzzy set (shortly IFS) on a set* X *is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0, 1])$ *is called the "degree of membership of* x *in* A*",* $\nu_A(x)$, $(\nu_A : X \to [0, 1])$ *is called the "degree of non-membership of* x *in* A*", and where* μ_A *and* ν_A *satisfy the following condition:*

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } \; x \in X.$$

The hesitation, indeterminacy, or uncertainty degree of x is defined by:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Definition 3 [2]. *Let* $A \in$ *IFS and let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *then the following set is called the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

Definition 4 [4]. *Let for brevity* $(a_{i,j})$ *denote a matrix with elements, denoted also by* a *and let* $M_{3\times3}(\mathbb{R})$ *be the set of* (3×3)*-matrices with elements – real numbers.*

Let X be a fixed set. Then Ω and Γ are defined as follows:

$$\Omega = \{\Theta \mid \Theta : \text{IFS}(X) \to \text{IFS}(X) \text{ is an IFMO}\}$$

$$\Gamma = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times 3}(\mathbb{R}) \;\&\; 0 \leq (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} \leq 1$$
$$\&\; 0 \leq a_{31}+a_{32} \leq 1\}.$$

Definition 5 [4]. *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in \text{IFS}(X)$.*

The mapping $\varphi_A : \Gamma \to \Omega$,

$$\varphi_A((a_{i,j})) = \{\langle x, a_{11}\mu_A(x) + a_{21}\nu_A(x) + a_{31}, a_{12}\mu_A(x) + a_{22}\nu_A(x) + a_{32}\rangle :$$

$$x \in X \;\&\; 0 \leq (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} + a_{31} + a_{32} \leq 1 \;\&\; 0 \leq a_{31}+a_{32} \leq 1\}.$$

After this, it is shown that the second type of IFMOs with matrices is as follows. Let a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, $a_{32} \in [-1, 1]$ satisfy the inequalities

$$0 \leq (\max)(\min)\{a_{11}+a_{12}, a_{21}+a_{22}\} + a_{31} + a_{32} \leq 1$$

and

$$0 \leq a_{31}+a_{32} \leq 1.$$

Then

$$\Theta(A) = \{\langle x, a_{11}\mu_A(x) + a_{21}\nu_A(x) + a_{31}, a_{12}\mu_A(x) + a_{22}\nu_A(x) + a_{32}\rangle : x \in X\}$$
$$= \begin{bmatrix} \mu_A(x) & \nu_A(x) & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

It is clear that for the present case, sets

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times 2}(\mathbb{R}) \;\&\; (a_{i,j}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\}$$

and

$$M_1 = \{(a_{i,j}) : (a_{i,j}) \in M_{3\times 3}(\mathbb{R}) \;\&\; (a_{i,j}) = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}\}$$

are equal. For brevity, in this paper, if $\varphi_A((a_{i,j})) = \Theta$, the following notation is used:

$$\Theta = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}.$$

2 Algebra of intuitionistic fuzzy modal operators

The algebraic properties of intuitionistic fuzzy modal operators were studied in [4]. In addition, in order to study the other algebraic properties of Intuitionistic fuzzy modal operators in an easy way, we use the matrix representation of Intuitionistic fuzzy modal operators. In this section, $1T$, $2T$ and UT represent respectively Type-1, Type-2 and Uni-Type modal operators. It can easily be seen that the matrix product of $1T$ and $2T$ ($2T$ and $1T$)sometimes gives $1T$ or $2T$. For example;(1) $\boxplus \in 1T$ and $G_{\alpha,\beta} \in 2T \ni \boxplus G_{\alpha,\beta} = \boxplus_{\frac{\alpha}{2},\frac{\beta}{2},\frac{\beta}{2}} \in 1T$ (the matrix product), (2) $G_{\alpha,\beta} \in 2T$ and $\boxplus \in 1T \ni G_{\alpha,\beta}\boxplus = \boxplus_{\frac{\alpha}{2},\frac{\beta}{2},\frac{1}{2}} \in 1T$ (the matrix product), $\boxplus \in 1T$ and $\Box \in 2T \ni \boxplus\Box(\Box\boxplus) = H^{*}_{\frac{1}{2},1} \in 2T$ (the matrix product). Also, it can be easily seen that the matrix product of $1T$ and UT (UT and $1T$) or $2T$ and UT (UT and $2T$) give $X_{a,b,c,d,e,f}$ which is the most generalized form of intuitionistic fuzzy modal operators. Now, we give the necessity conditions to find which type of modal operators we want when two Intuitionistic fuzzy modal operators are in matrix product.

Theorem 1. *Let $\Psi \in 2T$, then $\exists \Phi \in 1T \ni \Psi\Phi \in 2T$.*

Proof. Let $\Psi \in 2T$,

$$\Psi = \begin{bmatrix} b & c & 0 \\ d & e & 0 \\ f & g & 1 \end{bmatrix}$$

where: $b, c, d, e, f, g \in [0, 1]$ and

$(\max)(\min)(b + c, d + e) + f + g \in [0, 1]$ & $f + g \in [0, 1]$.

1. $h, i, j, k, l, m \in [0, 1]$
2. $kc = ei$
3. $hd = bj$
4. $h \leq b$
5. $k \leq e$
6. $\frac{h}{b} f \leq l$
7. $\frac{k}{e} g \leq m$
8. $l + m + (1 - g)\frac{k}{e} + (1 - f)\frac{h}{b} \leq 1$
9. $(\max)(\min)\left(\frac{h}{b}d + \frac{e}{c}i, \frac{b}{d}j + \frac{k}{e}c\right) + l + m \in [0, 1]$

$$\Phi = \begin{bmatrix} \frac{h}{b} & 0 & 0 \\ 0 & \frac{k}{e} & 0 \\ l - \frac{h}{b}f & m - \frac{k}{e}g & 1 \end{bmatrix}$$

(1) $h, b \in [0, 1]$ and by using 4.: $0 \leq \frac{h}{b} \leq 1$

(2) $k, e \in [0, 1]$ and by using 5.: $0 \leq \frac{k}{e} \leq 1$

(3) $f \in [0, 1] \implies \frac{h}{b}f \in [0, 1]$ and by using 6.: $0 \leq l - \frac{h}{b}f \leq 1$

(4) $g \in [0, 1] \implies \frac{k}{e}g \in [0, 1]$ and by using 7.: $0 \leq m - \frac{k}{e}g \leq 1$

(5) $\frac{h}{b} \leq \frac{k}{e}$

$$\frac{k}{e}+l-\frac{h}{b}f+m-\frac{k}{e}g=l+m+(1-g)\frac{k}{e}-\frac{h}{b}f$$

we can easily show the inequality below with the help of $6.$ and $7.$

$$\begin{aligned}
0 &\leq \frac{k}{e}+l+m-m-l-g\frac{k}{e} \\
&\leq l+m+(1-g)\frac{k}{e}-\frac{h}{b}f \\
&\leq l+m+(1-g)\frac{k}{e}-\frac{h}{b}f+\frac{h}{b} \\
&= l+m+(1-g)\frac{k}{e}+(1-f)\frac{h}{b}\leq 1
\end{aligned}$$

(6) $\frac{h}{b}\geq\frac{k}{e}$

$$\frac{h}{b}+l-\frac{h}{b}f+m-\frac{k}{e}g=l+m+(1-f)\frac{h}{b}-\frac{k}{e}g$$

we can write the following inequality with the help of $6.$ and $7.$

$$\begin{aligned}
0 &\leq \frac{h}{b}=l+m-l-m+\frac{h}{b} \\
&\leq l+m+(1-f)\frac{h}{b}-\frac{k}{e}g \\
&\leq l+m+(1-f)\frac{h}{b}-\frac{k}{e}g+\frac{k}{e} \\
&= l+m+(1-g)\frac{k}{e}+(1-f)\frac{h}{b}\leq 1
\end{aligned}$$

From the results above, we get that:
$(\max)\left(\frac{h}{b},\frac{k}{e}\right)+l-\frac{h}{b}f+m-\frac{k}{e}g\in[0,1] \implies \Phi\in 1T$

$$\begin{aligned}
\Psi\Phi &= \begin{bmatrix} b & c & 0 \\ d & e & 0 \\ f & g & 1 \end{bmatrix}\begin{bmatrix} \frac{h}{b} & 0 & 0 \\ 0 & \frac{k}{e} & 0 \\ l-\frac{h}{b}f & m-\frac{k}{e}g & 1 \end{bmatrix} \\
&= \begin{bmatrix} b\frac{h}{b} & c\frac{k}{e} & 0 \\ d\frac{h}{b} & e\frac{k}{e} & 0 \\ f\frac{h}{b}+l-\frac{h}{b}f & g\frac{k}{e}+m-\frac{k}{e}g & 1 \end{bmatrix} \\
&= \begin{bmatrix} h & c\frac{k}{e} & 0 \\ d\frac{h}{b} & k & 0 \\ l & m & 1 \end{bmatrix} \text{ and with the help of 2. and 3.} \\
&= \begin{bmatrix} h & i & 0 \\ j & k & 0 \\ l & m & 1 \end{bmatrix}
\end{aligned}$$

$$(*) \qquad h,i,j,k,l,m \in [0,1]$$

$$(**) \qquad h+i \leq j+k$$

$$0 \leq j+k+l+m = \frac{h}{b}d + \frac{e}{c}i + l + m \leq 1 \text{ (using 9.)}$$

$$0 \leq h+i+l+m = \frac{b}{d}j + \frac{k}{e}c + l + m \leq 1 \text{ (using 9.)}$$

$$(***) \qquad h+i \geq j+k$$

$$0 \leq h+i+l+m = \frac{b}{d}j + \frac{k}{e}c + l + m \leq 1 \text{ (using 9.)}$$

$$0 \leq j+k+l+m = \frac{h}{b}d + \frac{e}{c}i + l + m \leq 1 \text{ (using 9.)}$$

$$(****) \qquad 0 \leq \frac{h}{b}d + \frac{e}{c}i + l + m \leq 1 \implies 0 \leq l+m \leq 1$$

From the results above, we get that: $\Psi\Phi \in 2T$ $\square$

Example 1. *If we get* $\Psi = \square$ *and* $\Psi\Phi = H^*_{\frac{1}{2},1}$ *then*

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

So we find that: $\Phi = \boxplus$.

Theorem 2. *Let* $\Phi \in 1T$ *then* $\exists \Psi \in 2T \ni \Phi\Psi \in 2T$.

Proof. Let $\Phi \in 1T$,

$$\Phi = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & 1 \end{bmatrix}$$

where: $\alpha, \beta, \gamma, \delta \in [0,1]$ and $(\max)(\alpha, \beta) + \gamma + \delta \in [0,1]$.

1. $b,c,d,e,f,g \in [0,1]$
2. $b+c \leq \alpha$
3. $d+e \leq \beta$
4. $\frac{b}{\alpha}\gamma + \frac{d}{\beta}\delta \leq f$
5. $\frac{c}{\alpha}\gamma + \frac{e}{\beta}\delta \leq g$
6. $\beta(b+c) = \alpha(d+e)$
7. $f+g+(1-\delta)\left(\frac{d+e}{\beta}\right) + (1-\gamma)\left(\frac{b+c}{\alpha}\right) \leq 1$
8. $(\max)(\min)(\alpha,\beta) + f + g \in [0,1]$

$$\Psi = \begin{bmatrix} \frac{b}{\alpha} & \frac{c}{\alpha} & 0 \\ \frac{d}{\beta} & \frac{e}{\beta} & 0 \\ f - \frac{b}{\alpha}\gamma - \frac{d}{\beta}\delta & g - \frac{c}{\alpha}\gamma - \frac{e}{\beta}\delta & 1 \end{bmatrix}$$

$$b, \alpha \in [0,1] \text{ and by using 2.: } 0 \leq \frac{b}{\alpha} \leq 1 \tag{1}$$

$$c, a \in [0,1] \text{ and by using 2.: } 0 \leq \frac{c}{\alpha} \leq 1 \tag{2}$$

$$d, \beta \in [0,1] \text{ and by using 3.: } 0 \leq \frac{d}{\beta} \leq 1 \tag{3}$$

$$e, \beta \in [0,1]\text{and by using 3.: } 0 \leq \frac{e}{\beta} \leq 1 \tag{4}$$

$$\gamma \in [0,1] \implies \frac{b}{\alpha}\gamma \text{ and } \delta \in [0,1] \implies \frac{d}{\beta}\delta \in [0,1] \text{ and by using 4.:} \tag{5}$$

$$0 \leq f - \frac{b}{\alpha}\gamma - \frac{d}{\beta}\delta \leq f \leq 1$$

$$\gamma \in [0,1] \implies \frac{c}{\alpha}\gamma \text{ and } \delta \in [0,1] \implies \frac{e}{\beta}\delta \in [0,1] \text{ and by using 5.:} \tag{6}$$

$$0 \leq g - \frac{c}{\alpha}\gamma - \frac{e}{\beta}\delta \leq g \leq 1$$

$$\frac{b}{\alpha} + \frac{c}{\alpha} \leq \frac{d}{\beta} + \frac{e}{\beta} \tag{7}$$

$$\begin{aligned} \frac{d}{\beta}+\frac{e}{\beta}+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta &= f+g+(1-\delta)\frac{d}{\beta}+(1-\delta)\frac{e}{\beta}-\gamma(\frac{b+c}{\alpha}) \\ &= f+g+(1-\delta)(\frac{d+e}{\beta})-\gamma(\frac{b+c}{\alpha}) \end{aligned}$$

and $(\max)\,(\alpha,\beta)+\gamma+\delta \in [0,1] \implies \gamma+\delta \leq 1 \implies \gamma \leq 1-\delta$
From 6.

$$\begin{aligned} \frac{b+c}{\alpha} &= \frac{d+e}{\beta} \Rightarrow \gamma(\frac{b+c}{\alpha}) \leq (1-\delta)(\frac{d+e}{\beta}) \\ &\Rightarrow 0 \leq (1-\delta)(\frac{d+e}{\beta}) - \gamma(\frac{b+c}{\alpha}) \\ &\leq (1-\delta)(\frac{d+e}{\beta}) - \gamma(\frac{b+c}{\alpha}) + (\frac{b+c}{\alpha}) \\ &= (1-\delta)(\frac{d+e}{\beta}) + (1-\gamma)\,(\frac{b+c}{\alpha}) \\ &\leq f+g+(1-\delta)(\frac{d+e}{\beta}) + (1-\gamma)\,(\frac{b+c}{\alpha}) \leq 1 \end{aligned}$$

and

$$\begin{aligned}\frac{b}{\alpha}+\frac{c}{\alpha}+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta &= f+g+(1-\gamma)\,\frac{b}{\alpha}+(1-\gamma)\,\frac{c}{\alpha}-\delta(\frac{d+e}{\beta}) \\ &= f+g+(1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta})\end{aligned}$$

and $(\max)\,(\alpha,\beta)+\gamma+\delta\in[0,1] \implies \gamma+\delta\le 1 \implies \delta\le 1-\gamma$
From 6.

$$\begin{aligned}\frac{b+c}{\alpha} &= \frac{d+e}{\beta}\Rightarrow \delta(\frac{d+e}{\beta})\le(1-\gamma)\,(\frac{b+c}{\alpha}) \\ &\Rightarrow\ 0\le(1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta}) \\ &\le\ (1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta})+(\frac{d+e}{\beta}) \\ &=\ (1-\gamma)\,(\frac{b+c}{\alpha})+(1-\delta)\,(\frac{d+e}{\beta}) \\ &\le\ f+g+(1-\gamma)\,(\frac{b+c}{\alpha})+(1-\delta)\,(\frac{d+e}{\beta})\le 1\end{aligned}$$

$$\frac{b}{\alpha}+\frac{c}{\alpha}\ge\frac{d}{\beta}+\frac{e}{\beta} \tag{8}$$

$$\begin{aligned}\frac{b}{\alpha}+\frac{c}{\alpha}+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta &= f+g+(1-\gamma)\,\frac{b}{\alpha}+(1-\gamma)\,\frac{c}{\alpha}-\delta(\frac{d+e}{\beta}) \\ &= f+g+(1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta})\end{aligned}$$

From 6.

$$\begin{aligned}\frac{b+c}{\alpha} &= \frac{d+e}{\beta}\Rightarrow \delta(\frac{d+e}{\beta})\le(1-\gamma)\,(\frac{b+c}{\alpha}) \\ &\Rightarrow\ 0\le(1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta}) \\ &\le\ (1-\gamma)\,(\frac{b+c}{\alpha})-\delta(\frac{d+e}{\beta})+(\frac{d+e}{\beta}) \\ &=\ (1-\gamma)\,(\frac{b+c}{\alpha})+(1-\delta)\,(\frac{d+e}{\beta}) \\ &\le\ f+g+(1-\gamma)\,(\frac{b+c}{\alpha})+(1-\delta)\,(\frac{d+e}{\beta})\le 1\end{aligned}$$

and

$$\begin{aligned}\frac{d}{\beta}+\frac{e}{\beta}+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta &= f+g+(1-\delta)\frac{d}{\beta}+(1-\delta)\frac{e}{\beta}-\gamma(\frac{b+c}{\alpha}) \\ &= f+g+(1-\delta)\left(\frac{d+e}{\beta}\right)-\gamma(\frac{b+c}{\alpha})\end{aligned}$$

and $(\max)\,(\alpha,\beta)+\gamma+\delta\in[0,1] \implies \gamma+\delta\leq 1 \implies \gamma\leq 1-\delta$
From 6.

$$\begin{aligned}
\frac{b+c}{\alpha} &= \frac{d+e}{\beta} \implies \gamma(\frac{b+c}{\alpha}) \leq (1-\delta)(\frac{d+e}{\beta}) \\
&\Rightarrow\ 0 \leq (1-\delta)(\frac{d+e}{\beta}) - \gamma(\frac{b+c}{\alpha}) \\
&\leq\ (1-\delta)(\frac{d+e}{\beta}) - \gamma(\frac{b+c}{\alpha}) + (\frac{b+c}{\alpha}) \\
&=\ (1-\delta)(\frac{d+e}{\beta}) + (1-\gamma)\,(\frac{b+c}{\alpha}) \\
&\leq\ f+g+(1-\delta)(\frac{d+e}{\beta}) + (1-\gamma)\,(\frac{b+c}{\alpha}) \leq 1
\end{aligned}$$

From the results above, we get that:
$(\max)\,(\min)\left(\frac{b}{\alpha}+\frac{c}{\alpha},\frac{d}{\beta}+\frac{e}{\beta}\right)+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta\in[0,1] \implies \Psi\in 2T$

(3) $$\frac{b}{\alpha}\gamma+\frac{d}{\beta}\delta\leq f \text{ and } \frac{c}{\alpha}\gamma+\frac{e}{\beta}\delta\leq g$$

$$\implies 0\leq \frac{b}{\alpha}\gamma+\frac{d}{\beta}\delta+\frac{c}{\alpha}\gamma+\frac{e}{\beta}\delta\leq f+g \text{ and } (\max)\,(\min)\,(\alpha,\beta)+f+g\in[0,1]$$
$$\implies 0\leq f+g\leq 1$$

Now, let us prove that: $\Phi\Psi\in 2T$.

$$\begin{aligned}
\Phi\Psi &= \begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ \gamma & \delta & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\alpha} & \frac{c}{\alpha} & 0\\ \frac{d}{\beta} & \frac{e}{\beta} & 0\\ f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta & g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta & 1\end{bmatrix}\\
&= \begin{bmatrix}\alpha\frac{b}{\alpha} & \alpha\frac{c}{\alpha} & 0\\ \beta\frac{d}{\beta} & \beta\frac{e}{\beta} & 0\\ \gamma\frac{b}{\alpha}+\delta\frac{d}{\beta}+f-\frac{b}{\alpha}\gamma-\frac{d}{\beta}\delta & \gamma\frac{c}{\alpha}+\delta\frac{e}{\beta}+g-\frac{c}{\alpha}\gamma-\frac{e}{\beta}\delta & 1\end{bmatrix}\\
&= \begin{bmatrix}b & c & 0\\ d & e & 0\\ f & g & 1\end{bmatrix}
\end{aligned}$$

$(*)$ $b,c,d,e,f,g\in[0,1]$
$(**)$ $b+c\leq d+e$
$0\leq d+e+f+g\leq\beta+f+g\leq 1$ (using 8)
$0\leq b+c+f+g\leq\alpha+f+g\leq 1$ (using 8)
$(***)$ $b+c\geq d+e$
We can easily satisfy the conditions by using (*)
$(****)$ $0\leq f+g\leq f+g+(1-\delta)\left(\frac{d+e}{\beta}\right)+(1-\gamma)\left(\frac{b+c}{\alpha}\right)\leq 1 \implies f+g\in[0,1]$
From the results above, we can write that:
$(\max)\,(\min)\,(b+c,d+e)+f+g\in[0,1] \implies \Phi\Psi\in 2T$ □

Example 2. *If we get* $\Phi = \boxplus$ *and* $\Phi\Psi = H^*_{\frac{1}{2},1}$ *then*

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

So, we find that: $\Psi = \square$.

Theorem 3. *Let* $\Phi_1, \Phi_2 \in 1T$ *then* $\Phi_1\Phi_2 \in 1T$

Proof. Lets $\Phi_1 \in 1T$ and $\Phi_2 \in 1T$.

$$\Phi_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & 1 \end{bmatrix}$$

where: $\alpha, \beta, \gamma, \delta \in [0,1]$ and $(\max)(\alpha, \beta) + \gamma + \delta \in [0,1]$ and

$$\Phi_2 = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ d & e & 1 \end{bmatrix}$$

where: $b, c, d, e \in [0,1]$ and $(\max)(b, c) + d + e \in [0,1]$.

So:

$$\begin{aligned} \Phi_1\Phi_2 &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & 1 \end{bmatrix} \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ d & e & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha b & 0 & 0 \\ 0 & \beta c & 0 \\ \gamma b + d & \delta c + e & 1 \end{bmatrix} \end{aligned}$$

(1) $\alpha, b \in [0,1] \implies \alpha b \in [0,1]$
(2) $\beta, c \in [0,1] \implies \beta c \in [0,1]$ and $\gamma \in [0,1]$
(3) $(\max)(b, c) + d + e \in [0,1] \implies b + d \in [0,1]$
$\implies 0 \leq \gamma b \leq b \implies \gamma b + d \leq b + d \leq 1 \implies \gamma b + d \in [0,1]$
(4) $(\max)(b, c) + d + e \in [0,1] \implies c + e \in [0,1]$ and $\delta \in [0,1]$
$\implies 0 \leq \delta c \leq c \implies \delta c + e \leq c + e \leq 1 \implies \delta c + e \in [0,1]$
(5) $\alpha b \leq \beta c$
$\beta c + \gamma b + d + \delta c + e = d + e + (\beta + \delta) c + \gamma b$
By using: $b, c, d, e, \beta, \gamma, \delta \in [0,1]$ and $(\max)(\alpha, \beta) + \gamma + \delta \in [0,1]$
$\implies \beta + c \in [0,1]$ and $(\beta + \delta) c \leq c$ and $\gamma b \leq b$
$0 \leq d + e + (\beta + \delta) c + \gamma b \leq d + e + c + b \leq 1$
$\implies (\max)(\alpha b, \beta c) + \gamma b + d + \delta c + e \in [0,1]$
(6) $\alpha b \geq \beta c$
$\alpha b + \gamma b + d + \delta c + e = d + e + (\alpha + \gamma) b + \delta c$

By using: $b, c, d, e, \gamma, \delta \in [0,1]$ and $(\max)(\alpha, \beta) + \gamma + \delta \in [0,1]$
$\implies \gamma + \delta \in [0,1]$ and $(\alpha + \gamma) b \leq b$ and $\delta c \leq c$
$0 \leq d + e + (\alpha + \gamma) b + \delta c \leq d + e + b + c \leq 1$
$\implies (\max)(\alpha b, \beta c) + \gamma b + d + \delta c + e \in [0,1]$
From the results above, we find that: $\Phi_1 \Phi_2 \in 1T$. $\square$

Example 3. *$\boxplus$, $\boxplus_{\alpha,\beta,\gamma} \in 1T$ and $\boxplus\boxplus_{\alpha,\beta,\gamma} = \boxplus_{\frac{\alpha}{2},\frac{\beta}{2},\frac{\beta}{2}+\gamma}$.*

Theorem 4. *Let $\Psi_1, \Psi_2 \in 2T$ then $\Psi_1 \Psi_2 \in 2T$*

Proof. Let $\Psi_1 \in 2T$ and $\Psi_2 \in 2T$.

$$\Psi_1 = \begin{bmatrix} b & c & 0 \\ d & e & 0 \\ f & g & 1 \end{bmatrix}$$

where: $b, c, d, e, f, g \in [0,1]$ and
$(\max)(\min)(b + c, d + e) + f + g \in [0,1], f + g \in [0,1]$.

$$\Psi_2 = \begin{bmatrix} x & y & 0 \\ z & t & 0 \\ w & q & 1 \end{bmatrix}$$

where: $x, y, z, t, w, q \in [0,1]$ and
$(\max)(\min)(x + y, z + t) + w + q \in [0,1], w + q \in [0,1]$.
So:

$$\begin{aligned} \Psi_1 \Psi_2 &= \begin{bmatrix} b & c & 0 \\ d & e & 0 \\ f & g & 1 \end{bmatrix} \begin{bmatrix} x & y & 0 \\ z & t & 0 \\ w & q & 1 \end{bmatrix} \\ &= \begin{bmatrix} bx + cz & by + ct & 0 \\ dx + ez & dy + et & 0 \\ fx + gz + w & fy + gt + q & 1 \end{bmatrix} \end{aligned}$$

(1) By using: $b, c, x, z \in [0,1]$ and $(\max)(\min)(b + c, d + e) + f + g \in [0,1]$
$b + c \in [0,1]$ and $bx \leq b, cz \leq c \implies 0 \leq bx + cz \implies b + c \leq 1$
$\implies bx + cz \in [0,1]$
(2) By using: $b, c, y, t \in [0,1]$ and $(\max)(\min)(b + c, d + e) + f + g \in [0,1]$
$b + c \in [0,1]$ and $by \leq b, ct \leq c \implies 0 \leq by + ct \leq b + c \leq 1$
$\implies by + ct \in [0,1]$
(3) By using: $d, e, x, z \in [0,1]$ and $(\max)(\min)(b + c, d + e) + f + g \in [0,1]$
$d + e \in [0,1]$ and $dx \leq d, ez \leq e \implies 0 \leq dx + ez \leq d + e \leq 1$
$\implies dx + ez \in [0,1]$
(4) By using: $d, e, y, t \in [0,1]$ and $(\max)(\min)(b + c, d + e) + f + g \in [0,1]$
$d + e \in [0,1]$ and $dy \leq d, et \leq e \implies 0 \leq dy + et \leq d + e \leq 1$
$\implies dy + et \in [0,1]$

(5) By using: $f, g, x, z, w \in [0,1]$ and $(\max)\,(\min)\,(x+y, z+t)+w+q \in [0,1]$
$x+z+w \in [0,1]$ and $fx \leq x, gz \leq z \implies 0 \leq fx+gz+w \leq x+z+w \leq 1$
$\implies fx+gz+w \in [0,1]$
(6) By using: $f, g, y, t, q \in [0,1]$ and $(\max)\,(\min)\,(x+y, z+t)+w+q \in [0,1]$
$y+t+q \in [0,1]$ and $fy \leq y, gt \leq t \implies 0 \leq fy+gt+q \leq y+t+q \leq$
$\implies fy+gt+q \in [0,1]$
(7) By using $fx+gz+w \in [0,1]\,, fy+gt+q \in [0,1]$ and $(\max)\,(\min)\,(x+y, z+t)+w+q \in [0,1]$
$x+y \in [0,1]$ and $z+t \in [0,1]$ and $f\,(x+y) \leq x+y, g\,(z+t) \leq z+t$

$$\begin{aligned} 0 &\leq fx+gz+w+fy+gt+q \\ &= f\,(x+y)+g\,(z+t)+w+q \leq x+y+z+t+w+q \leq 1 \\ &\Rightarrow fx+gz+w+fy+gt+q \in [0,1] \end{aligned}$$

To prove that:
$(\max)\,(\min)\,(bx+cz+by+ct, dx+ez+dy+et)+fx+gz+w+fy+gt+q \in [0,1]$
(8) $bx+cz+by+ct \leq dx+ez+dy+et$
By using: $(\max)\,(\min)\,(b+c, d+e)+f+g \in [0,1] \implies d+f \in [0,1]$ and $e+g \in [0,1]$
and $(d+f)\,(x+y) \leq x+y$ and $(e+g)\,(z+t) \leq z+t$
and $(\max)\,(\min)\,(x+y, z+t)+w+q \in [0,1]$

$$\begin{aligned} 0 &\leq dx+ez+dy+et+fx+gz+w+fy+gt+q \\ &= d\,(x+y)+e\,(z+t)+f\,(x+y)+g\,(z+t)+w+q \\ &= (d+f)\,(x+y)+(e+g)\,(z+t)+w+q \\ &\leq x+y+z+t+w+q \in [0,1] \end{aligned}$$

So, $(\max)\,(bx+cz+by+ct, dx+ez+dy+et)+fx+gz+w+fy+gt+q \in [0,1]$
$(\max)\,(\min)\,(b+c, d+e)+f+g \in [0,1] \implies b+f \in [0,1]$ and $c+g \in [0,1]$
and $(b+f)\,(x+y) \leq x+y$, $(c+g)\,(z+t) \leq z+t$ and
$(\max)\,(\min)\,(x+y, z+t)+w+q \in [0,1]$

$$\begin{aligned} 0 &\leq bx+cz+by+ct+fx+gz+w+fy+gt+q \\ &= b\,(x+y)+c\,(z+t)+f\,(x+y)+g\,(z+t)+w+q \\ &= (b+f)\,(x+y)+(c+g)\,(z+t)+w+q \\ &\leq x+y+z+t+w+q \leq 1 \end{aligned}$$

So, $(\min)\,(bx+cz+by+ct, dx+ez+dy+et)+fx+gz+w+fy+gt+q \in [0,1]$
$(9) bx+cz+by+ct \leq dx+ez+dy+et$
to prove conditions for conditions above, we follow the same way of (8)
All results proved above show us that: $\Psi_1\Psi_2 \in 2T$. $\square$

Example 4. *$F_{\alpha,\beta}$, $\Diamond \in 2T$ and $F_{\alpha,\beta}\Diamond = D_{1-\beta}$*

Corollary 1. *Let $N \in \{1, 2, U\}$. If $\Omega, \Sigma \in NT$, then $\Omega\Sigma \in NT$.*

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