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(k_1, k_2)-Intuitionistic fuzzy sets

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Abstract : In this paper we introduce definitions of (k_1, k_2)-intuitionistic fuzzy set, (k_1, k_2)-intuitionistic fuzzy relation and study some of their properties.

Keywords : Fuzzy subset, intuitionistic fuzzy set, (k_1, k_1)-intuitionistic fuzzy set, fuzzy relation.

0. Introduction

Let X be a nonempty set. In [1] and [2], Atanassov introduced the idea of intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ where $\mu_A : X \to I$ (the closed interval [0, 1]) and $\nu_A : X \to I$ are such that $\mu_A(x) + \nu_A(x) \leq 1,\ \forall\ x \in X$. Later on Atanassov himself and many other authors including us did lot of works on intuitionistic fuzzy setting. For references see [4], [6], [7], [8], [9] etc.

In the class of intuitionistic fuzzy sets (IFSs) we observe that for any IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\},\ 0 \leq \mu_A(x) \wedge \nu_A(x) \leq 0.5,\ \forall\ x \in X$. So it is interesting to study the class of IFSs $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ having the property

$$k_1 \leq \mu_A(x) \wedge \nu_A(x) \leq k_2,\ \forall\ x \in X$$

where $k_1,\ k_2$ are two constants lying in [0, 0.5].
This problem is proposed by K. T. Atanassov in a personal communication with us.
In this paper, we study such a class of IFSs with the above property and establish some results.

In Section 1, we define (k_1, k_2)-intuitionistic fuzzy sets ($(k_1,\ k_2)$-IFSs) and study some of its properties.
In Section 2, we define (k_1, k_2)-intuitionistic fuzzy relations ((k_1, k_2)-IFRs) and study various properties of (k_1, k_2)-IFRs.

1. (k_1, k_2)-Intuitionistic fuzzy sets

Definition 1.1 Let X be a nonempty set. A (k_1, k_2)-intuitionistic fuzzy set ((k_1, k_2)-IFS)

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \qquad (*)$$

is an IFS on X satisfying the property

$$k_1 \leq \mu_A(x) \wedge \nu_A(x) \leq k_2,\ \forall\ x \in X \qquad (**)$$

where $k_1, k_2 \in [0, 0.5]$ are two constants.
The property (**) will be called (k_1, k_2)-condition.

For simplicity, we shall use $A = (\mu_A, \nu_A)$ in place of (*).
The collection of all (k_1, k_2)-IFSs on X is denoted by $\mathcal{C}_0(X)$.

Basic algebraic operations on $\mathcal{C}_0(X)$:

Let $A, B, A_i \in \mathcal{C}_0(X),\ \forall\ i \in I$. Then inclusion, equality, complementation, arbitrary union and arbitrary intersection on $\mathcal{C}_0(X)$ are defined as follows :
(1) $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x),\ \forall\ x \in X$,

(2) $A = B \Leftrightarrow A \subset B$ and $B \subset A$,
(3) $\bar{A} = (\nu_A, \mu_A)$,
(4) $\cup_i A_i = (\vee_i \mu_{A_i}, \wedge_i \nu_{A_i})$,
(5) $\cap_i A_i = (\wedge_i \mu_{A_i}, \vee_i \nu_{A_i})$.

Note 1.2 It is to be noted that the definitions given above are consistent as the (k_1, k_2)-condition is satisfied where it is required.

Definition 1.3 The smallest and the greatest elements of $\mathcal{C}_0(X)$, denoted, respectively, by $S = (\mu_S, \nu_S)$ and $G = (\mu_G, \nu_G)$, are defined by $\mu_S(x) = k_1,\ \nu_S(x) = 1 - k_1,\ \forall\ x \in X$ and $\mu_G(x) = 1 - k_1,\ \nu_G(x) = k_1,\ \forall\ x \in X$, respectively.

Remark 1.4 $\mathcal{C}_0(X)$ is a complete sublattice of the lattice of all IFSs with S and G being, respectively, its smallest and greatest element.

Definition 1.5 We define $\tilde{0} = (0, \tilde{1})$ and $\tilde{1} = (1, \tilde{0})$.

Definition 1.6 An intuitionistic fuzzy point P on X is an IFS such that $\exists$ an $x \in X$ satisfying $\mu_P(x) > 0$ and $\mu_P(y) = 0,\ \nu_P(y) = 1,\ \forall\ y(\neq x) \in X$.

Remark 1.7 In order that $\tilde{0}$ or $\tilde{1} \in \mathcal{C}_0(X)$, we must have $k_1 = 0$. Further, $k_1 \neq 0 \Rightarrow \wedge_{x \in X}(\mu_A(x) \wedge \nu_A(x)) \geq k_1 \neq 0 \Rightarrow \mu_A(x) \neq 0$ and $\nu_A(x) \neq 0,\ \forall x \in X$.
So, if $k_1 \neq 0$, then no intuitionistic fuzzy point is a (k_1, k_2)-IFS.

Definition 1.8 A (k_1, k_2)-IFS P on X is said to be a (k_1, k_2)- intuitionistic fuzzy point ((k_1, k_2)-IFP) on X if $\exists\ x \in X$ such that
$\nu_P(x) < 1 - k_1$
and
$\mu_P(y) = k_1,\ \nu_P(y) = 1 - k_1,\ \forall\ y(\neq x) \in X$.

Such a (k_1, k_2)-IFP is denoted by P_x. If for a (k_1, k_2)-IFP $P_x,\ \mu_P(x) = a$ and $\nu_P(x) = b$, then P_x is also denoted by $(a, b)_x$.

Let $A \in \mathcal{C}_0(X)$. A (k_1, k_2)-IFP P_x is said to belong to A if $\mu_P(x) \leq \mu_A(x)$ and $\nu_P(x) \geq \nu_A(x)$. This is denoted by $P_x \tilde{\in} A$.

Theorem 1.9 *$A = \cup\{P_x\ ;\ P_x \tilde{\in} A\}$, where $A \in \mathcal{C}_0(X)$.*

Proof. If $\{P_x\ ;\ P_x \tilde{\in} A\} = \phi$, then $\mu_A(x) = k_1$ and $\nu_A(x) = 1 - k_1,\ \forall\ x \in X$; for, otherwise $\exists\ \xi \in X$ such that $\nu_A(\xi) < 1 - k_1$. Defining a (k_1, k_2)-IFP P_ξ by $\mu_P(\xi) = \mu_A(\xi),\ \nu_P(\xi) = \nu_A(\xi)$, we see that $P_\xi \tilde{\in} A$, a contradiction.
So, if $\{P_x\ ;\ P_x \tilde{\in} A\} = \phi$, then $\cup\{P_x\ ;\ P_x \tilde{\in} A\} = S = A$.
Next, suppose $\{P_x\ ;\ P_x \tilde{\in} A\} \neq \phi$. Put $\cup\{P_x\ ;\ P_x \tilde{\in} A\} = Q$. Clearly $Q \supset S$. Take $\xi \in X$. Consider the following two cases :
Case I : $\nu_A(\xi) = 1 - k_1$,
Case II : $\nu_A(\xi) < 1 - k_1$.
In case I, $\mu_A(\xi) = k_1$. So, $\mu_A(\xi) = \mu_S(\xi) \leq \mu_Q(\xi)$ and $\nu_A(\xi) = \nu_S(\xi) \geq \nu_Q(\xi)$ i.e., $\mu_A(\xi) \leq \mu_Q(\xi)$ and $\nu_A(\xi) \geq \nu_Q(\xi)$.
In case II, choose a (k_1, k_2)-IFP P_ξ such that $\mu_P(\xi) = \mu_A(\xi),\ \nu_P(\xi) = \nu_A(\xi)$. Then $P_\xi \tilde{\in} A$ and $\mu_A(\xi) = \mu_P(\xi) \leq \mu_Q(\xi)$ and $\nu_A(\xi) = \nu_P(\xi) \geq \nu_Q(\xi)$.
Thus considering both the cases I and II we see that
$\mu_A(\xi) \leq \mu_Q(\xi)$ and $\nu_A(\xi) \geq \nu_Q(\xi),\ \forall\ \xi \in X$.
Thus $A \subset Q = \cup\{P_x\ ;\ P_x \tilde{\in} A\}$. Obviously $Q \subset A$.
Hence $A = Q = \cup\{P_x\ ;\ P_x \tilde{\in} A\}$.

Corollary 1.10 *For two (k_1, k_2)-IFSs A and B, $A = B$ iff $P_x \tilde{\in} A \Leftrightarrow P_x \tilde{\in} B$.*

Theorem 1.11 *For all $A, B, C, B_i \in \mathcal{C}_0(X),\ i \in I$, we have*
(1) $S \subset A \subset G$,
(2) $\bar{S} = G;\ \bar{G} = S$,

(3) $A \subset B$ *and* $B \subset C \Rightarrow A \subset C$,
(4) $A, B \subset A \cup B;\ A, B \supset A \cap B$,
(5) $A \cup B = B \cup A;\ A \cap B = B \cap A$,
(6) $A \cup (B \cup C) = (A \cup B) \cup C;\ A \cap (B \cap C) = (A \cap B) \cap C$,
(7) $A \cup (\cap_i B_i) = \cap_i (A \cup B_i);\ A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$,
(8) $A \subset B \Leftrightarrow \bar{A} \supset \bar{B}$,
(9) $\overline{(\cup_i B_i)} = \cap_i \bar{B}_i;\ \overline{(\cap_i B_i)} = \cup_i \bar{B}_i$,
(10) $\bar{\bar{A}} = A$.

The proof is straightforward.

Definition 1.12 Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Let $A \in \mathcal{C}_0(X)$. Then the image of A, under f, denoted by $f(A) = (\mu_{f(A)}, \nu_{f(A)})$, is defined by

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ k_1, & \text{otherwise} \end{cases}$$

$$\nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 1 - k_1, & \text{otherwise} \end{cases}$$

Let $B \in \mathcal{C}_0(Y)$. Then the preimage of B, under f, denoted by $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$, is defined by
$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)),\ \nu_{f^{-1}(B)}(x) = \nu_B(f(x))$.

Theorem 1.13 *Let* $A, A_i \in \mathcal{C}_0(X)$ *and* $B, B_j \in \mathcal{C}_0(Y),\ i \in I, j \in J$ *and* $f : X \to Y$ *be a mapping. Then*
(a) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$,
(b) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,
(c) $f(\bar{A}) \supset \overline{[f(A)]}$, *if* f *is surjective,*
(d) $f^{-1}(\bar{B}) = \overline{[f^{-1}(B)]}$,
(e) $A \subset f^{-1}(f(A))$, *the equality holds if f is injective,*
(f) $f(f^{-1}(B)) \subset B$, *the equality holds if f is surjective,*
(g) $f^{-1}(\cup_j B_j) = \cup_j f^{-1}(B_j)$,
(h) $f^{-1}(\cap_j B_j) = \cap_j f^{-1}(B_j)$,
(i) $f(\cup_i A_i) = \cup_i f(A_i)$,
(j) $f(\cap_i A_i) \subset \cap_i f(A_i)$, *the equality holds if f is injective,*
(k) If $g : Y \to Z$ *be a mapping then* $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$, *for any* $C \in \mathcal{C}_0(Z)$, *where* gof *is the composition of* g *and* f.

The proof is straightforward.

2. (k_1, k_2)-Intuitionistic fuzzy relations

Intuitionistic fuzzy relation was studied by Atanassov himself in [3] and then by Bustince et al. in [5]. In this Section, we give the definition of (k_1, k_2)-intuitionistic fuzzy relation and study some of its properties.
Let X, Y and Z be three ordinary nonempty sets.

Definition 2.1 A (k_1, k_2)-intuitionistic fuzzy relation $R = \{< (x, y), \mu_R(x, y), \nu_R(x, y) > : x \in X,\ y \in Y\}$ where $\mu_R : X \times Y \to I,\ \nu_R : X \times Y \to I$ is an IFS on $X \times Y$ satisfying the property

$$k_1 \leq \mu_R(x, y) \wedge \nu_R(x, y) \leq k_2,\ \forall\ (x, y) \in X \times Y$$

where $k_1, k_2 \in [0, 0.5],\ k_1 \leq k_2$, are two constants.

The collection of all (k_1, k_2)-IFRs on $X \times Y$ is denoted by $\mathcal{C}_0(X \times Y)$.

Definition 2.2 Let $R \in \mathcal{C}_0(X \times Y)$. Then we define inverse relation of R, denoted by R^{-1}, by

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y),\ \nu_{R^{-1}}(y,x) = \nu_R(x,y),\ \forall\ (x,y) \in X \times Y.$$

Definition 2.3 Let $P, Q, P_i \in \mathcal{C}_0(X \times Y), i \in I$. Then for every $(x,y) \in X \times Y$ we define
(a) $P \leq Q \Leftrightarrow \mu_P(x,y) \leq \mu_Q(x,y)$ and $\nu_P(x,y) \geq \nu_Q(x,y)$,
(b) $P \prec Q \Leftrightarrow \mu_P(x,y) \leq \mu_Q(x,y)$ and $\nu_P(x,y) \leq \nu_Q(x,y)$,
(c) $\cup_i P_i = \{< \vee_i \mu_{P_i}(x,y), \wedge_i \nu_{P_i}(x,y) > :\ (x,y) \in X \times Y\}$,
(d) $\cap_i P_i = \{< \wedge_i \mu_{P_i}(x,y), \vee_i \nu_{P_i}(x,y) > :\ (x,y) \in X \times Y\}$,
(e) $\bar{P} = \{< \nu_P(x,y), \mu_P(x,y) > :\ (x,y) \in X \times Y\}$.

Theorem 2.4 *Let* $P, Q, R \in \mathcal{C}_0(X \times Y)$. *Then*
(a) $P \leq Q \Rightarrow P^{-1} \leq Q^{-1};\ P \prec Q \Rightarrow P^{-1} \prec Q^{-1}$,
(b) $P \leq Q \Rightarrow \bar{Q} \leq \bar{P};\ P \prec Q \Rightarrow \bar{P} \prec \bar{Q}$,
(c) $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$,
(d) $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$,
(e) $(P^{-1})^{-1} = P$,
(f) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R);\ P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$,
(g) $P \cup Q \geq P, Q;\ P \cap Q \leq P, Q$ *and* $P, Q \not\prec P \cup Q,\ P \cap Q \not\prec P, Q$,
(h) $P \geq Q,\ P \geq R \Rightarrow P \geq Q \cup R;\ P \leq Q,\ P \leq R \Rightarrow P \leq Q \cap R$,
(i) $P \succ Q,\ P \succ R \Rightarrow P \succ Q \cup R;\ P \prec Q,\ P \prec R \Rightarrow P \prec Q \cap R.$

The proof is straightforward.

Note 2.5 Let $R \in \mathcal{C}_0(X \times Y)$ and $P \in \mathcal{C}_0(Y \times Z)$. Then

$$k_1 \leq [\vee_y \{\mu_R(x,y) \wedge \mu_P(y,z)\}] \wedge [\wedge_y \{\nu_R(x,y) \vee \nu_P(y,z)\}] \leq k_2.$$

Proof. We have,

$$\begin{aligned}
&[\vee_{y'}\{\mu_R(x,y') \wedge \mu_P(y',z)\}] \wedge [\nu_R(x,y) \vee \nu_P(y,z)]\\
&\qquad \geq [\mu_R(x,y) \wedge \mu_P(y,z)] \wedge [\nu_R(x,y) \vee \nu_P(y,z)]\\
&\qquad = [\mu_R(x,y) \wedge \mu_P(y,z) \wedge \nu_R(x,y)] \vee [\mu_R(x,y) \wedge \mu_P(y,z) \wedge \nu_p(y,z)]\\
&\qquad = [\{\mu_R(x,y) \wedge \nu_R(x,y)\} \wedge \mu_P(y,z)] \vee [\mu_R(x,y) \wedge \{\mu_P(y,z) \wedge \nu_P(y,z)\}]\\
&\qquad \geq [k_1 \wedge \mu_P(y,z)] \vee [\mu_R(x,y) \wedge k_1]\\
&\qquad = k_1 \vee k_1,\ \text{ since } k_1 \leq \mu_P(y,z) \wedge \nu_P(y,z) \leq \mu_P(y,z); \text{ similarly } k_1 \leq \mu_R(x,y)\\
&\qquad = k_1,\ \forall\ y \in Y.
\end{aligned}$$

Hence

$$[\vee_y \{\mu_R(x,y) \wedge \mu_P(y,z)\}] \wedge [\wedge_y \{\nu_R(x,y) \vee \nu_P(y,z)\}] \geq k_1. \tag{1}$$

$$\begin{aligned}
&[\mu_R(x,y) \wedge \mu_P(y,z)] \wedge [\wedge_{y'}\{\nu_R(x,y') \vee \nu_P(y',z)\}]\\
&\qquad \leq [\mu_R(x,y) \wedge \mu_P(y,z)] \wedge [\nu_R(x,y) \vee \nu_P(y,z)]\\
&\qquad = [\mu_R(x,y) \wedge \mu_P(y,z) \wedge \nu_R(x,y)] \vee [\mu_R(x,y) \wedge \mu_P(y,z) \wedge \nu_p(y,z)]\\
&\qquad = [\{\mu_R(x,y) \wedge \nu_R(x,y)\} \wedge \mu_P(y,z)] \vee [\mu_R(x,y) \wedge \{\mu_P(y,z) \wedge \nu_P(y,z)\}]\\
&\qquad \leq [k_2 \wedge \mu_P(y,z)] \vee [\mu_R(x,y) \wedge k_2]\\
&\qquad \leq k_2 \vee k_2\\
&\qquad = k_2,\ \forall\ y \in Y.
\end{aligned}$$

Hence

$$[\vee_y \{\mu_R(x,y) \wedge \mu_P(y,z)\}] \wedge [\wedge_y \{\nu_R(x,y) \vee \nu_P(y,z)\}] \leq k_2. \tag{2}$$

(1) and (2) give the required result.

Note 2.6 Let $R \in \mathcal{C}_0(X \times Y)$ and $P \in \mathcal{C}_0(Y \times Z)$. Then

$$k_1 \leq [\wedge_y \{\mu_R(x,y) \vee \mu_P(y,z)\}] \wedge [\vee_y \{\nu_R(x,y) \wedge \nu_P(y,z)\}] \leq k_2.$$

The proof is analogous to that of Note 2.5.

Now we shall define two kinds of composition of (k_1, k_2)-IFRs.

Definition 2.7 Let $R \in \mathcal{C}_0(X \times Y)$ and $P \in \mathcal{C}_0(Y \times Z)$. Then we define composed relation on $X \times Z$, denoted by PoR, by

$$PoR = \{< (x,z), \mu_{PoR}(x,z), \nu_{PoR}(x,z) >: x \in X, z \in Z\}.$$

where
$\mu_{PoR}(x,z) = \vee_y\{\mu_R(x,y) \wedge \mu_P(y,z)\},\ \nu_{PoR}(x,z) = \wedge_y\{\nu_R(x,y) \vee \nu_P(y,z)\}.$

Definition 2.8 Let $R \in \mathcal{C}_0(X \times Y)$ and $P \in \mathcal{C}_0(Y \times Z)$. Then we define another composed relation on $X \times Z$, denoted by $P * R$, by

$$P * R = \{< (x,z), \mu_{P*R}(x,z), \nu_{P*R}(x,z) >: x \in X, z \in Z\}.$$

where
$\mu_{P*R}(x,z) = \wedge_y\{\mu_R(x,y) \vee \mu_P(y,z)\},\ \nu_{P*R}(x,z) = \vee_y\{\nu_R(x,y) \wedge \nu_P(y,z)\}.$

Definition 2.9 Let $P, R \in \mathcal{C}_0(X \times X)$. Then P and R are said to commute if $PoR = RoP$.

We now give the following Theorems 2.10 - 2.12, whose proofs are similar to the corresponding Theorems in [5].

Theorem 2.10 *For* $R \in \mathcal{C}_0(X \times Y),\ P \in \mathcal{C}_0(Y \times Z),\ (PoR)^{-1} = R^{-1}oP^{-1}$ *holds.*

Theorem 2.11 *If* $R, R_i \in \mathcal{C}_0(X \times Y)$ *and* $P, P_i \in \mathcal{C}_0(Y \times Z),\ i = 1, 2,$ *then*
$(a)\ P_1 \leq P_2 \Rightarrow P_1oR \leq P_2oR,$
$(b)\ R_1 \leq R_2 \Rightarrow PoR_1 \leq PoR_2,$
$(c)\ P_1 \prec P_2 \Rightarrow P_1oR \prec P_2oR,$
$(d)\ R_1 \prec R_2 \Rightarrow PoR_1 \prec PoR_2,$
$(e)\ If\ R, P \in GR(X \times X)$ *and* $P \leq R$ *then* $PoP \leq RoR$.

Theorem 2.12 *For* $R \in \mathcal{C}_0(X \times Y),\ Q \in \mathcal{C}_0(Y \times Z)$ *and* $P \in \mathcal{C}_0(Z \times U)$
$(PoQ)oR = Po(QoR)$ *holds.*

Theorem 2.13 *For each* $R \in \mathcal{C}_0(X \times Y)$ *and* $P_i \in \mathcal{C}_0(Y \times Z), i \in I,$
$(a)\ (\cup_i P_i)oR = \cup_i(P_ioR),$
$(b)\ (\cap_i P_i)oR = \cap_i(P_ioR)$
holds.

Proof.

$$\begin{aligned}
\mu_{((\cup_i P_i)oR)}(x,z) &= \vee_y\{\mu_R(x,y) \wedge \mu_{\vee_i P_i}(y,z)\} \\
&= \vee_y\{\mu_R(x,y) \wedge (\vee_i \mu_{P_i}(y,z))\} \\
&= \vee_y\{\vee_i(\mu_R(x,y) \wedge \mu_{P_i}(y,z))\} \\
&= \vee_i\{\vee_y(\mu_R(x,y) \wedge \mu_{P_i}(y,z))\} \\
&= \vee_i \mu_{P_ioR}(x,z),\ \forall\ (x,z) \in X \times Z.
\end{aligned}$$

Similarly, it can be shown that $\nu_{((\cup_i P_i)oR)}(x,z) = \vee_i \nu_{P_ioR}(x,z),\ \forall\ (x,z) \in X \times Z.$
Hence $(\cup_i P_i)oR = \cup_i(P_ioR)$.

(b) The proof is analogous to that of (a).

Here we shall define reflexivity, antireflexivity and study some of their properties.

Definition 2.14 Let $R \in \mathcal{C}_0(X \times X)$. Then
(a) R is reflexive of type-1 if

$$\mu_R(x,x) = 1 - k_1,\ \forall\ x \in X.$$

(b) R is reflexive of type-2 if

$$\mu_R(x,x) \wedge \mu_R(y,y) \geq k_2 \vee \mu_R(x,y),\ \forall\ x,y \in X,\ \ \nu_R(x,x) = k_1,\ \forall\ x \in X.$$

(c) R is reflexive of type-3 if

$$\mu_R(x,x) \wedge \mu_R(y,y) \geq \mu_R(x,y),\ \ \nu_R(x,x) \vee \nu_R(y,y) \leq \nu_R(x,y),\ \forall\ x,y \in X.$$

Definition 2.15 Let $R \in \mathcal{C}_0(X \times X)$. Then
(a) R is antireflexive of type-1 if

$$\nu_R(x,x) = 1 - k_1,\ \forall\ x \in X.$$

(b) R is antireflexive of type-2 if

$$\mu_R(x,x) = k_1,\ \forall\ x \in X,\ \ \nu_R(x,x) \wedge \nu_R(y,y) \geq k_2 \vee \nu_R(x,y),\ \forall\ x,y \in X.$$

(c) R is antireflexive of type-3 if

$$\mu_R(x,x) \vee \mu_R(y,y) \leq \mu_R(x,y),\ \ \nu_R(x,x) \wedge \nu_R(y,y) \geq \nu_R(x,y),\ \forall\ x,y \in X.$$

Remark 2.16
(a) Reflexivity (antireflexivity) of type-1 $\Rightarrow$ reflexivity (antireflexivity) of type-2 $\Rightarrow$ reflexivity (antireflexivity) of type-3. It can be easily shown by constructing examples that the reverse implication does not hold.

Theorem 2.17
(a) If $R \in \mathcal{C}_0(X \times X)$ is reflexive of any type then $R \leq RoR$,
(b) If $R \in \mathcal{C}_0(X \times X)$ is antireflexive of any type then $R \geq R * R$.

Proof. (a)

$$\begin{aligned}
\mu_{RoR}(x,z) &= \vee_y\{\mu_R(x,y) \wedge \mu_R(y,z)\} \\
&= (\mu_R(x,x) \wedge \mu_R(x,z)) \vee (\vee_{y \neq x}(\mu_R(x,y) \wedge \mu_R(y,z))) \\
&= \mu_R(x,z) \vee (\vee_{y \neq x}(\mu_R(x,y) \wedge \mu_R(y,z))),\ \text{for any type of reflexivity } R \\
&\geq \mu_R(x,z).
\end{aligned}$$

$$\begin{aligned}
\nu_{RoR}(x,z) &= \wedge_y\{\nu_R(x,y) \vee \nu_R(y,z)\} \\
&= (\nu_R(x,x) \vee \nu_R(x,z)) \wedge (\wedge_{y \neq x}(\nu_R(x,y) \vee \nu_R(y,z))) \\
&= \nu_R(x,z) \wedge (\wedge_{y \neq x}(\nu_R(x,y) \vee \nu_R(y,z))),\ \text{for any type of reflexivity } R \\
&\leq \nu_R(x,z).
\end{aligned}$$

Hence $R \leq RoR$.

(b) The proof is similar to that of (a).

Next we give examples of (k_1,k_2)-IFRs which satisfy the property $R \leq RoR$ ($R \geq R * R$), but R is not reflexive (antireflexive) of any type.

Example 2.18 (a) Let $X = \{a,b,c\}$ and $R \in \mathcal{C}_0(X \times X)$ with $k_1 = 0,\ k_2 = 0.4$ is given by

$$\mu_R = \begin{pmatrix} & a & b & c \\ a & 0.4 & 0.6 & 0.1 \\ b & 0.6 & 0.7 & 0.6 \\ c & 0.1 & 0 & 0.3 \end{pmatrix},\quad \nu_R = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.4 & 0.9 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.5 & 0.3 & 0.6 \end{pmatrix}$$

Therefore $RoR \in \mathcal{C}_0(X \times X)$ is given by

$$\mu_{RoR} = \begin{pmatrix} & a & b & c \\ a & 0.6 & 0.6 & 0.6 \\ b & 0.6 & 0.7 & 0.6 \\ c & 0.1 & 0.1 & 0.3 \end{pmatrix}, \quad \nu_{RoR} = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.4 & 0.4 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

This gives that $R \le RoR$, but R is not reflexive of any type.

(b) Let $X = \{a, b, c\}$ and $R \in \mathcal{C}_0(X \times X)$ with $k_1 = 0.3,\ k_2 = 0.4$ is given by

$$\mu_{R} = \begin{pmatrix} & a & b & c \\ a & 0.4 & 0.6 & 0.3 \\ b & 0.6 & 0.7 & 0.6 \\ c & 0.3 & 0.4 & 0.3 \end{pmatrix}, \quad \nu_{R} = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.3 & 0.6 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.5 & 0.5 & 0.6 \end{pmatrix}$$

Therefore $R * R \in \mathcal{C}_0(X \times X)$ is given by

$$\mu_{R*R} = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.4 & 0.3 \\ b & 0.6 & 0.6 & 0.6 \\ c & 0.3 & 0.4 & 0.3 \end{pmatrix}, \quad \nu_{R*R} = \begin{pmatrix} & a & b & c \\ a & 0.5 & 0.5 & 0.6 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.5 & 0.5 & 0.6 \end{pmatrix}$$

This gives that $R \ge R * R$, but R is not antireflexive of any type.

Theorem 2.19 *Let $R, R_1, R_2 \in \mathcal{C}_0(X \times X)$. Then*
(a) If R is reflexive (antireflexive) of any type, then RoR $(R * R)$ is reflexive (antireflexive) of the same type,
(b) If R is reflexive (antireflexive) of any type, then R^{-1} is reflexive (antireflexive) of the same type.
(c) If both $R_j,\ j \in J$ is reflexive (antireflexive) of any type, then $\cap_j R_j$ $(\cup_j R_j)$ is reflexive (antireflexive) of the same type.
(d) If R_j is reflexive (antireflexive) of the type-1 for some $j = j_0 \in J$then $\cup_j R_j$ $(\cap_j R_j)$ is so. If $R_j,\ j \in J$ is reflexive (antireflexive) of the type-i, then $\cup_j R_j$ $(\cap_j R_j)$ is reflexive (antireflexive) of the type-i, $i = 2$ and 3.

Proof. (a) If R is reflexive of any type, then

$$\begin{aligned} \mu_{RoR}(x,x) &= \vee_y\{\mu_R(x,y) \wedge \mu_R(y,x)\} \\ &= \{\mu_R(x,x) \wedge \mu_R(x,x)\} \vee \{\vee_{y\neq x}(\mu_R(x,y) \wedge \mu_R(y,x))\} \\ &= \mu_R(x,x) \vee \{\vee_{y\neq x}(\mu_R(x,y) \wedge \mu_R(y,x))\} \\ &= \mu_R(x,x), \forall\ x \in X \qquad (1) \end{aligned}$$

Similarly, $\nu_{RoR}(x,x) = \nu_R(x,x), \forall\ x \in X$ (2)

If R is reflexive of the type-2 or 3, then we have, for $y \neq x$

$$\begin{aligned} \mu_{RoR}(x,y) &= \vee_z\{\mu_R(x,z) \wedge \mu_R(z,y)\} \\ &= \{\mu_R(x,x) \wedge \mu_R(x,y)\} \vee \{\mu_R(x,y) \wedge \mu_R(y,y)\} \vee \{\vee_{z\neq x, z\neq y}(\mu_R(x,z) \\ \wedge\mu_R(z,y))\} &= \mu_R(x,y) \vee \{\mu_R(x,y) \wedge \mu_R(y,y)\} \vee \{\vee_{z\neq x, z\neq y}(\mu_R(x,z) \wedge \mu_R(z,y))\} \\ &\le \mu_R(x,x) \wedge \mu_R(y,y),\ \text{since } \mu_R(x,x) \wedge \mu_R(y,y) \ge \mu_R(x,y), \forall\ x,y \in X \\ &= \mu_{RoR}(x,x) \wedge \mu_{RoR}(y,y),\ \text{by (1)} \qquad (3) \end{aligned}$$

If R is reflexive of type-2, then
$\mu_{RoR}(x,x) = \mu_R(x,x) \geq k_2$. Similarly $\mu_{RoR}(y,y) \geq k_2$. So $\mu_{RoR}(x,x) \wedge \mu_{RoR}(y,y) \geq k_2$. Therefore,

$$k_2 \vee \mu_{RoR}(x,y) \leq \mu_{RoR}(x,x) \wedge \mu_{RoR}(y,y). \tag{4}$$

If R is reflexive of type-3, then we have, for $y \neq x$

$$\begin{aligned}
\nu_{RoR}(x,y) &= \wedge_z\{\nu_R(x,z) \vee \nu_R(z,y)\} \\
&= \{\nu_R(x,x) \vee \nu_R(x,y)\} \wedge \{\nu_R(x,y) \vee \nu_R(y,y)\} \wedge \{\wedge_{z\neq x, z\neq y}(\nu_R(x,z) \\
&\quad \vee \nu_R(z,y))\} \\
&= \nu_R(x,y) \wedge \{\nu_R(x,y) \vee \nu_R(y,y)\} \wedge \{\wedge_{z\neq x, z\neq y}(\nu_R(x,z) \vee \nu_R(z,y))\} \\
&\geq \nu_R(x,x) \vee \nu_R(y,y), \text{ since } \nu_R(x,x) \vee \nu_R(y,y) \leq \nu_R(x,y), \forall\, x,y \in X \\
&= \nu_{RoR}(x,x) \vee \nu_{RoR}(y,y), \text{ by (2)} \qquad (5)
\end{aligned}$$

Hence by (1) - (5), RoR is reflexive of the type as that of R.
The proof for antireflexivity is similar to that of reflexivity.

(b) The proof is straightforward.

(c) Let $P = \cap_{j\in J} R_j$. Then $\mu_P(x,x) = \wedge_j \mu_{R_j}(x,x),\ \forall\ x \in X$. Hence, P is of type-1 reflexive if R_j is so, $\forall\ j \in J$.
Next, for $y \neq x$

$$\begin{aligned}
\mu_P(x,y) &= \wedge_j \mu_{R_j}(x,y) \\
&\leq \wedge_j(\mu_{R_j}(x,x) \wedge \mu_{R_j}(y,y)) \\
&= (\wedge_j \mu_{R_j}(x,x)) \wedge (\wedge_j \mu_{R_j}(y,y)) \\
&= \mu_P(x,x) \wedge \mu_P(y,y).
\end{aligned}$$

For reflexive of type-2,
$\mu_P(x,x) = \wedge_j \mu_{R_j}(x,x) \geq k_2$, (since $\mu_{R_j}(x,x) \geq k_2,\ j \in J$). Similarly $\mu_P(y,y) \geq k_2$. So $\mu_P(x,x) \wedge \mu_P(y,y) \geq k_2$. Therefore, $k_2 \vee \mu_P(x,y) \leq \mu_P(x,x) \wedge \mu_P(y,y)$.
Similarly, we can establish the properties of ν_P.
Hence, from above, P is reflexive of the type as of $R_j,\ j \in J$.
The proof for antireflexivity is similar to that of reflexivity.

(d) The proof is similar to that of (c) and is therefore omitted.

As regard to the Theorem 2.19(d), we give an example to show that for the reflexivity of the type-2 and 3, the condition that only one of R_1 and R_2 is reflexive does not imply the reflexivity of $R_1 \cup R_2$.

Example 2.20 Let $X = \{a,b\}$ and $R_1,\ R_2 \in \mathcal{C}_0(X \times X)$ with $k_1 = 0,\ k_2 = 0.5$ are given by

$$\mu_{R_1} = \begin{pmatrix} & a & b \\ a & 0.5 & 0.4 \\ b & 0.5 & 0.7 \end{pmatrix},\ \nu_{R_1} = \begin{pmatrix} & a & b \\ a & 0 & 0.6 \\ b & 0.5 & 0 \end{pmatrix}$$

and

$$\mu_{R_2} = \begin{pmatrix} & a & b \\ a & 0.4 & 0.6 \\ b & 0.5 & 0.7 \end{pmatrix},\ \nu_{R_2} = \begin{pmatrix} & a & b \\ a & 0.6 & 0.3 \\ b & 0.5 & 0.3 \end{pmatrix}$$

We see that R_1 is reflexive of type-2 and type-3 but R_2 is not so and $R_1 \cup R_2$ is not reflexive of type-2 or type-3.

Now we define symmetric relation and study some of its properties.

Definition 2.21 A relation $R \in \mathcal{C}_0(X \times X)$ is called symmetric if $R = R^{-1}$ i.e., if for all $(x, y) \in X \times X,\ \mu_R(x, y) = \mu_R(y, x),\ \nu_R(x, y) = \nu_R(y, x)$.

Theorem 2.22

(a) If $P, R \in \mathcal{C}_0(X \times X)$ are symmetric, then $PoR = (RoP)^{-1}$,

(b) If R is symmetric then RoR is symmetric.

Proof. (a) Since $R = R^{-1}$ and $P = P^{-1}$, therefore, $PoR = P^{-1}oR^{-1} = (RoP)^{-1}$.

(b) The proof is obvious.

Note 2.23 In general the composition of two symmetric relations may not be symmetric. However, the composition of two symmetric relations is symmetric iff they commute.

Lastly, we define transitive and c-transitive (k_1, k_2)-IFRs and study some of their properties.

Definition 2.24 $R \in \mathcal{C}_0(X \times X)$ is said to be transitive (c-transitive) if $R \geq RoR$ ($R \leq R*R$).

Definition 2.25 Let $R \in \mathcal{C}_0(X \times X)$.

(a) The transitive closure of R is defined to be the minimum (k_1, k_2)-IFR $\hat{R}$ on $X \times X$ which contains R and it is transitive, that is to say

(1) $R \leq \hat{R}$,

(2) $\hat{R}o\hat{R} \leq \hat{R}$,

(3) if $P \in \mathcal{C}_0(X \times X),\ R \leq P$ and P is transitive, then $\hat{R} \leq P$.

(b) The c-transitive closure of R is defined to be the biggest c-transitive relation $\breve{R} \in \mathcal{C}_0(X \times X)$ contained in R.

Notation 2.26 We denote $R^1 = R,\ R^n = RoRo...n$ times, $n \geq 2$ and $R^{*1} = R,\ R^{*n} = R * R * ...n$ times, $n \geq 2$.

Theorem 2.27 *For every $R \in \mathcal{C}_0(X \times X)$, it is verified that :*

(a) $\hat{R} = R^1 \cup R^2 \cup R^3 \cup ... \cup R^n \cup ... = \cup_{i=1}^{\infty} R^i$,

(b) $\breve{R} = R^{*1} \cap R^{*2} \cap R^{*3} \cap ... \cap R^{*n} \cap ... = \cap_{i=1}^{\infty} R^{*i}$.

Proof.

(a) (1) $R \leq \hat{R}$ is evident.

(2) Now we will use the distributive property of the composition w.r.t. '$\cup$'.

$\hat{R}o\hat{R} = (\cup_{i=1}^{\infty} R^i)o(\cup_{i=1}^{\infty} R^i) = \cup_{i=2}^{\infty} R^i \leq \cup_{i=1}^{\infty} R^i = \hat{R}$.

(3) Now we will find the minimum transitive relation which contains R.

Let us take $R \leq P,\ P$ being transitive, that is $P^2 \leq P$. Now using Theorem 2.11(e), we get

$R \leq P$,

then, $R^2 \leq P^2 \leq P$,

similarly, $R^3 \leq P$,

. . .

therefore, $\cup_{n=1}^{\infty} R^n \leq P \Rightarrow \hat{R} \leq P$.

Hence $\hat{R} = \cup_{i=1}^{\infty} R^i$.

(b) The proof is analogous to that of (a).

Theorem 2.28 *Let $R,\ P \in \mathcal{C}_0(X \times X)$. Then $R \leq P \Rightarrow \hat{R} \leq \hat{P}$ and $\breve{R} \geq \breve{P}$.*

The proof is straightforward.

Corollary 2.29 *For every $R \in \mathcal{C}_0(X \times X),\ \ \breve{R} \leq R \leq \hat{R}$ holds.*

Corollary 2.30

(1) If $R \in \mathcal{C}_0(X \times X)$ is reflexive of any type and transitive then $R = RoR$.

(2) If $R \in \mathcal{C}_0(X \times X)$ is antireflexive of any type and c-transitive then $R = R * R$.

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