Notes on Intuitionistic Fuzzy Sets
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A note on mean value and dispersion of intuitionistic fuzzy events

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Abstract: In this paper, we compare two definitions of mean value and dispersion for intuitionistic fuzzy events. We show the connection between these two definitions and we introduce some types of mean values induced by intuitionistic fuzzy state and by intuitionistic fuzzy probability.
Keywords: Intuitionistic fuzzy event, Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Intuitionistic fuzzy probability, Intuitionistic fuzzy mean value, Intuitionistic fuzzy dispersion, Intuitionistic fuzzy distribution function.
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1 Introduction

In connection with the studying of the limit theorems for the intuitionistic fuzzy observables there was a need to define the notion of mean value and dispersion. First, the notion of integrable intuitionistic fuzzy observable appeared in the paper [8]. There, the authors used the probability measures as a composition an intuitionistic fuzzy observable and intuitionistic fuzzy probability for definition. Later, B. Riečan defined an intuitionistic mean value as an integral with help of an intuitionistic fuzzy distribution function, see [14]. In this paper, we define the intuitionistic fuzzy mean value and the intuitionistic fuzzy dispersion with the help of intuitionistic fuzzy state and we show the connection between these two definitions. We study the intuitionistic fuzzy mean value in connection with the intuitionistic fuzzy probability, too.

We note that in the whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 Preliminary notions

In this section, we present the basic notions like an intuitionistic fuzzy event, an intuitionistic fuzzy state, an intuitionistic fuzzy observable and an intuitionistic fuzzy probability. Recall that the notion of intuitionistic fuzzy sets was introduced by K. T. Atanassov in 1983 as a generalization of Zadeh's fuzzy sets (see [1, 2, 3]).

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$. The function $\mu_A : \Omega \longrightarrow [0,1]$ will be called the membership function and the function $\nu_A : \Omega \longrightarrow [0,1]$ will be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned}\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),\\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))\end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

The second basic notion in the probability theory is the notion of a state (see [14]).

Definition 2.3. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$*;*

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$*, then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$*;*

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$*,* $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets. Its elements are called Borel sets.

Definition 2.4. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1_\Omega - x^\sharp(A)\big)$ *for each* $A \in \mathcal{B}(R)$, *then* $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ *are observables, where* $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - measurable\}$.

Similarly as in the classical case the following theorem can be proved ([8, 14]).

Theorem 2.5. *Let* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be an IF-observable,* $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ *be an IF-state. Define the mapping* $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *is a probability measure.*

Proof. In [8] *Proposition 3.1.* □

Sometimes we need to work with an intuitionistic fuzzy probability, see [9, 10].

Definition 2.6. *Let* $\mathcal{F}$ *be the family of all IF-events in* Ω. *A mapping* $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ *is called an IF-probability, if the following conditions hold:*

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$;

(ii) *if* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, *then* $\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$;

(iii) *if* $\mathbf{A}_n \nearrow \mathbf{A}$, *then* $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.
(Recall that $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ *means that* $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$, *but* $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ *means* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$.*)*

Of course, each $\mathcal{P}(\mathbf{A})$ is an interval, denote it by $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$. By this way we obtain two real functions

$$\mathcal{P}^\flat : \mathcal{F} \to [0,1], \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$$

and some properties of $\mathcal{P}$ can be characterized by some properties of $\mathcal{P}^\flat, \mathcal{P}^\sharp$, see [11].

Theorem 2.7. *Let* $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ *and* $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ *for each* $\mathbf{A} \in \mathcal{F}$. *Then* $\mathcal{P}$ *is an IF-probability if and only if* $\mathcal{P}^\flat$ *and* $\mathcal{P}^\sharp$ *are IF-states.*

Proof. In [11] *Theorem 2.3.* □

3 Mean value using an intuitionistic fuzzy state

In this section we show two definitions of an intuitionistic fuzzy mean value and compare them. In the first case we are inspired by Kolmogorov case. If $\xi : \Omega \to R$ is a random variable, then

$$E(\xi) = \int_{\Omega} \xi \, dP = \int_R t \, dP_{\xi}(t),$$

where

$$P_{\xi}(B) = P(\xi^{-1}(B)).$$

Since now $\mathbf{m}_x : \mathcal{B}(R) \to [0,1]$ plays an analogous role as $P_{\xi} : \mathcal{B}(R) \to [0,1]$, we can define **intuitionistic fuzzy mean value** $\mathbf{E}(x)$ by the same formula.

Definition 3.1. *We say that an IF-observable x is an integrable IF-observable, if the integral $\int_R t \, d\mathbf{m}_x(t)$ exists. In this case we define IF-mean value*

$$\mathbf{E}(x) = \int_R t \, d\mathbf{m}_x(t).$$

If the integral $\int_R t^2 \, d\mathbf{m}_x(t)$ exists, then we define IF-dispersion $\mathbf{D}^2(x)$ by the formula

$$\mathbf{D}^2(x) = \int_R t^2 \, d\mathbf{m}_x(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \, d\mathbf{m}_x(t).$$

In [14] B. Riečan studied an IF-mean value with the help of an IF-distribution function. If $x : \mathcal{B}(R) \to \mathcal{F}$ is an IF-observable, and $\mathbf{m} : \mathcal{F} \to [0,1]$ is an IF-state, then the IF**-distribution function** of x is the function $\mathbf{F} : R \to [0,1]$ defined by the formula

$$\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$$

for each $t \in R$.

Similarly as in the classical case the following theorem can be proved ([14]).

Theorem 3.2. *Let $\mathbf{F} : R \to [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$. Then $\mathbf{F}$ is non-decreasing on R, left continuous in each point $t \in R$ and*

$$\lim_{t \to -\infty} \mathbf{F}(t) = 0, \ \lim_{t \to \infty} \mathbf{F}(t) = 1.$$

Proof. In [14] *Proposition 3.2.* □

The following theorem says about a connection between the definition of IF-mean value and IF-dispersion in *Definition 3.1* and the notion of IF-distribution function.

Theorem 3.3. *Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. Then*

$$\begin{aligned} \mathbf{E}(x) &= \int_R t \, d\mathbf{F}(t), \\ \mathbf{D}^2(x) &= \int_R t^2 \, d\mathbf{F}(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \, d\mathbf{F}(t). \end{aligned}$$

Proof. Since $\mathbf{F}$ is the IF-distribution function of the probability distribution $\mathbf{m}_x$, we have

$$\lambda_{\mathbf{F}}\big([a,b)\big) = \mathbf{F}(b) - \mathbf{F}(a) = \mathbf{m}_x\big([a,b)\big),$$

hence

$$\lambda_{\mathbf{F}} = \mathbf{m}_x.$$

Therefore,

$$\int_R t\, d\mathbf{F}(t) = \int_R t\, d\lambda_{\mathbf{F}}(t) = \int_R t\, d\mathbf{m}_x(t) = \mathbf{E}(x).$$

Similarly the other equality can be obtained. □

Therefore, we can define IF-mean value and IF-dispersion of an IF-observable in another way, see [4].

Definition 3.4. *Let* $\mathbf{F} : R \longrightarrow [0,1]$ *be the IF-distribution function of an IF-observable* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. *If there exists* $\int_R t\, d\mathbf{F}(t)$, *then we define the IF-mean value of* x *by the formula*

$$\mathbf{E}(x) = \int_R t\, d\mathbf{F}(t).$$

Moreover, if there exists $\int_R t^2\, d\mathbf{F}(t)$, *then we define the IF-dispersion* $\mathbf{D}^2(x)$ *by the formula*

$$\mathbf{D}^2(x) = \int_R t^2\, d\mathbf{F}(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2\, d\mathbf{F}(t).$$

In the classical case, we sometimes work with random variable, which is the composition of a measurable function and a random variable. In this case, the mean value is defined as follows

$$E(g \circ \xi) = \int_\Omega g \circ \xi\, dP = \int_R g\, dF,$$

where F is the distribution function of a random variable ξ and g is a measurable function. In [14] B. Riečan defined intuitionistic fuzzy mean value for this case as follows

Definition 3.5. *Let* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be an IF-observable,* $\mathbf{m} : \mathcal{F} \to [0,1]$ *be an IF-state,* $g : R \to R$ *be a measurable function,* $\mathbf{F} : R \longrightarrow [0,1]$ *be the IF-distribution function of* x *(i.e.* $\mathbf{F} = \mathbf{m} \circ x$*). Then we define the IF-mean value of* $g \circ x$ *by the formula*

$$\mathbf{E}(g \circ x) = \int_R g\, d\mathbf{F},$$

if the integral exists.

4 Mean value using an intuitionistic fuzzy probability

In this section, we explain the definition of an IF-mean value with the help of an IF-probability. Since the IF-probability can be decomposed on two IF-states (see *Theorem 2.7*), we can use the results from the previous section.

First, the notion of integrable observable in the sense of IF-probability appeared in [8]. There $\mathcal{P}_x^\flat = \mathcal{P}^\flat \circ x$ and $\mathcal{P}_x^\sharp = \mathcal{P}^\sharp \circ x$.

Definition 4.1. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. An IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ is called integrable, if there exist*

$$\mathbf{E}^\flat(x) = \int_R t\, d\mathcal{P}^\flat_x(t),\ \mathbf{E}^\sharp(x) = \int_R t\, d\mathcal{P}^\sharp_x(t).$$

Later, in [6] was defined the notion of a square integrable IF-observable and the notion of IF-dispersion and IF-mean value in the sense of IF-probability.

Definition 4.2. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states $\big($i.e. $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$, $\mathbf{A} \in \mathcal{F}\big)$ and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. We say that IF-observable x is integrable, if the integrals $\int_R t\, d\mathcal{P}^\flat_x(t)$, $\int_R t\, d\mathcal{P}^\sharp_x(t)$ exist. Then the IF-mean values are defined by*

$$\mathbf{E}^\flat(x) = \int_R t\, d\mathcal{P}^\flat_x(t) \quad , \quad \mathbf{E}^\sharp(x) = \int_R t\, d\mathcal{P}^\sharp_x(t).$$

We say that IF-observable x is square integrable, if the integrals $\int_R t^2\, d\mathcal{P}^\flat_x(t)$, $\int_R t^2\, d\mathcal{P}^\sharp_x(t)$ exist. Then the IF-dispersions are defined by

$$\begin{aligned} \mathbf{D}^2_\flat(x) &= \int_R (t - \mathbf{E}^\flat(x))^2\, d\mathcal{P}^\flat_x(t) = \int_R t^2\, d\mathcal{P}^\flat_x(t) - \big(\mathbf{E}^\flat(x)\big)^2, \\ \mathbf{D}^2_\sharp(x) &= \int_R (t - \mathbf{E}^\sharp(x))^2\, d\mathcal{P}^\sharp_x(t) = \int_R t^2\, d\mathcal{P}^\sharp_x(t) - \big(\mathbf{E}^\sharp(x)\big)^2. \end{aligned}$$

***Remark* 4.3.** *By Definition 4.2 the IF-mean value (or IF-dispersion) of IF-observable x induced by IF-probability consists of two IF-mean values (or IF-dispersions) induced by IF-state, i.e.*

$$\mathbf{E}(x) = [\mathbf{E}^\flat(x), \mathbf{E}^\sharp(x)] \text{ and } \mathbf{D}^2(x) = [\mathbf{D}^2_\flat(x), \mathbf{D}^2_\sharp(x)].$$

Similarly, we can define an IF-distribution function induced by an IF-probability. There instead of an IF-state $\mathbf{m}$ is used IF-probability $\mathcal{P}$, see [5].

Definition 4.4. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. Then a mapping $\mathbf{F} : R \to \mathcal{J}$ defined by formula*

$$\mathbf{F}(t) = \mathcal{P} \circ x((-\infty, t)) = \big[\mathcal{P}^\flat\big((x(-\infty, t))\big), \mathcal{P}^\sharp\big((x(-\infty, t))\big)\big] = [\mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)]$$

for each $t \in R$ is called IF- distribution function, where $\mathbf{F}^\flat, \mathbf{F}^\sharp : R \to [0, 1]$ are the corresponding distribution functions.

Theorem 4.5. *For fixed IF-probability $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ define $\mathbf{F}^\flat : R \to [0, 1], \mathbf{F}^\sharp : R \to [0, 1]$ by the formulas*

$$\mathbf{F}^\flat(t) = \mathcal{P}^\flat\Big(x\big((-\infty, t)\big)\Big),\ \mathbf{F}^\sharp(t) = \mathcal{P}^\sharp\Big(x\big((-\infty, t)\big)\Big).$$

Then $\mathbf{F}^\flat, \mathbf{F}^\sharp$ are distribution functions, and

$$\mathbf{E}^\flat(x) = \int_R t\, d\mathbf{F}^\flat(t) \quad , \quad \mathbf{E}^\sharp(x) = \int_R t\, d\mathbf{F}^\sharp(t),$$

$$\mathbf{D}^2_\flat(x) = \int_R (t - \mathbf{E}^\flat(x))^2\, d\mathbf{F}^\flat(t) \quad , \quad \mathbf{D}^2_\sharp(x) = \int_R (t - \mathbf{E}^\sharp(x))^2\, d\mathbf{F}^\sharp(t).$$

Proof. It follows from *Definition 4.2* and *Theorem 3.3*. □

Definition 4.6. *Let* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be an IF-observable,* $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ *be an IF-probability,* $g : R \to R$ *be a measurable function,* $\mathbf{F} : R \longrightarrow \mathcal{J}$ *be the IF-distribution function of* x *(i.e.* $\mathbf{F} = \mathcal{P} \circ x = [\mathbf{F}^\flat, \mathbf{F}^\sharp]$*). Then we define the IF-mean values of* $g \circ x$ *by the formulas*

$$\mathbf{E}^\flat(g \circ x) = \int_R g \, d\mathbf{F}^\flat, \ \mathbf{E}^\sharp(g \circ x) = \int_R g \, d\mathbf{F}^\sharp$$

if the integrals exist.

***Remark* 4.7.** *In this case the IF-mean value of IF-observable* $g \circ x$ *induced by IF-probability consists of two IF-mean values induced by IF-state, i.e.*

$$\mathbf{E}(g \circ x) = [\mathbf{E}^\flat(g \circ x), \mathbf{E}^\sharp(g \circ x)].$$

Conclusion

The paper concerns the probability theory on intuitionistic fuzzy events. We studied an intuitionistic fuzzy mean value and an intuitionistic fuzzy dispersion in connection with the intuitionistic fuzzy state and the intuitionistic fuzzy probability. We showed the relation between two ways of defining the intuitionistic fuzzy mean value and the intuitionistic fuzzy dispersion for the intuitionistic fuzzy observable.

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