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On Zadeh's intuitionistic fuzzy disjunction and conjunction

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1 Introduction

During the last ten years a lot of operations were defined over Intuitionistic Fuzzy Sets (IFSs; see [3]). Here, we will discuss three operations, generated by Zadeh's implication, introduced in fuzzy set theory (see, e.g., [8]). Its IFS-analogous was introduced in [5, 6] and here, on its basis, we will construct Zadeh's conjunction and disjunction.

In [8] 10 different fuzzy implications are discussed. Having in mind that in the classical logic the equality

$$x \vee y = \neg x \to y, \tag{1}$$

where x and y are logical variables, $\vee$ - disjunction, $\to$ - implication and $\neg$ - negation, we see that for any implication we can construct a disjunction and after this, using De Morgan's laws - a conjunction (or opposite).

2 Definition and algebraic properties of Zadeh's intuitionistic fuzzy disjunction and conjunction

The intuitionistic fuzzy propositional calculus has been introduced more than 20 years ago (see, e.g., [1, 3]). In it, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are the degrees of validity and of non-validity of x and there the following definitions are given.

Below we shall assume that for the two variables x and y the equalities: $V(x) = \langle a, b \rangle, V(y) = \langle c, d \rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For two variables x and y operations "conjunction" (&), "disjunction" ($\vee$), "implication" ($\rightarrow$), and "(standard) negation" ($\neg$) are defined by:

$$V(x\&y) = \langle \min(a, c), \max(b, d)\rangle,$$

$$V(x \vee y) = \langle \max(a, c), \min(b, d)\rangle,$$

$$V(x \rightarrow y) = \langle \max(b, c), \min(a, d)\rangle,$$

$$V(\neg x) = \langle b, a\rangle.$$

In [4] the following two operations, which are analogues to operations "conjunction" and "disjunction", are defined

$$V(x + y) = \langle a, b\rangle + \langle c, d\rangle = \langle a + c - ac, bd\rangle,$$

$$V(x.y) = \langle a, b\rangle.\langle c, d\rangle = \langle ac, b + d - bd\rangle.$$

The two standard modal operators (see [7]) have the following intuitionistic fuzzy estimations (see [2]).

$$V(\Box p) = \Box V(p) = \langle \mu(p), 1 - \mu(p)\rangle,$$

$$V(\Diamond p) = \Diamond V(p) = \langle 1 - \nu(p), \nu(p)\rangle.$$

Now, using (1) and intuitionistic fuzzy form of Zadeh's implicarion, introduced by the author in [5, 6] with the form

$$V(x \rightarrow_Z y) = \langle \max(b, \min(a, c)), \min(a, d)\rangle,$$

we will introduce a disjunction with the following form of its estimation

$$V(x \vee_Z y) = \langle a, b\rangle \vee_Z \langle c, d\rangle = \langle \max(a, \min(b, c)), \min(b, d)\rangle.$$

We will call the new disjunction *"Zadeh's intuitionistic fuzzy disjunction".*

We see also, that

$$V(x \rightarrow'_Z y) = \neg\langle a, b\rangle \vee_Z \langle c, d\rangle$$

$$= \langle b, a\rangle \vee_Z \langle c, d\rangle = \langle \max(b, \min(a, c)), \min(a, d)\rangle = V(x \rightarrow_Z y),$$

i.e., the implication generates a disjunction that generates the initial implication.

Let us suppose below that De Morgan's laws are valid, i.e.,

$$x\&y = \neg(\neg x \vee \neg y). \tag{2}$$

We must note immediately, that in IFS theory there are a lot of examples in which (2) is not valid, but this will be object of discussions in future research.

Therefore, using (2) and definition of $\vee_Z$, we can construct

$$V(x \wedge_Z y) = \langle a, b\rangle \wedge_Z \langle c, d\rangle = \langle \min(a, c), \max(b, \min(a, d))\rangle.$$

We will call the new conjunction *"Zadeh's intuitionistic fuzzy conjunction".*

For both new operations, having in mind that $\wedge_Z$ is obtained from $\vee_Z$ by (2), we will check fistly that

$$V(\neg(\neg x \wedge_Z \neg y)) = \neg(\neg\langle a,b\rangle \wedge_Z \neg\langle c,d\rangle))$$
$$= \neg(\langle b,a\rangle \wedge_Z \langle d,c\rangle)) = \neg\langle \min(b,d), \max(a, \min(b,c))\rangle$$
$$= \langle \max(a, \min(b,c)), \min(b,d)\rangle = V(x \vee_Z y).$$

Therefore, both operations are correctly defined one about the other.

We can check immediately the validity of equalities

$$V(x \wedge_Z x) = V(x),$$
$$V(x \vee_Z x) = V(x),$$

i.e. the Idempotent Laws hold, but equalities

$$V(x \wedge_Z y) = V(y \wedge_Z x),$$
$$V(x \vee_Z y) = V(y \vee_Z x),$$
$$V((x \wedge_Z y) \wedge_Z z) = x \wedge_Z (y \wedge_Z z)),$$
$$V((x \vee_Z y) \vee_Z z) = x \vee_Z (y \vee_Z z)),$$
$$V((x \wedge_Z y) \vee_Z z) = (x \vee_Z z) \wedge_Z (y \vee_Z z),$$
$$V((x \vee_Z y) \wedge_Z z) = (x \wedge_Z z) \vee_Z (y \wedge_Z z)$$

are not valid. For example, if $V(x) = \langle 0.0, 0.5\rangle, V(y) = \langle 0.0, 1.0\rangle$, then

$$V(x \wedge_Z y) = \langle 0.0, 0.5\rangle \neq \langle 0.0, 1.0\rangle = V(y \wedge_Z x).$$

Therefore, both operations are not commutative and associative ones and none is distributive with respect to the other.

In [1] the following relation is introduced for every $a, b, c, d \in [0,1]$ so that $a+b, c+d \in [0,1]$:

$$\langle a,b\rangle \leq \langle c,d\rangle \text{ if and only if } a \leq c \text{ and } d \geq b,$$
$$\langle a,b\rangle \geq \langle c,d\rangle \text{ if and only if } \langle c,d\rangle \leq \langle a,b\rangle.$$

The following inequalities are valid:

$$V(x.y) \leq V(x\&y) \leq V(x \wedge_Z y),$$
$$V(x \vee_Z y) \leq V(x \vee y) \leq V(x+y).$$

Theorem The following inequalities are valid

$$\text{(a) } V(\Box(x \vee_Z y)) \leq V(\Box x \vee_Z \Box y),$$
$$\text{(b) } V(\Diamond(x \wedge_Z y)) \geq V(\Diamond x \wedge_Z \Diamond y).$$

Proof. (a) Let x and y are two variables. Then

$$V(\Box(x \vee_Z y)) = \Box(\langle a,b\rangle \vee_Z \langle c,d\rangle)$$
$$= \Box\langle \max(a, \min(b,c)), \min(b,d)\rangle$$
$$= \langle \max(a, \min(b,c)), 1 - \max(a, \min(b,c))\rangle$$
$$\leq \langle \max(a, \min(1-a, c)), 1 - \max(a,c)\rangle$$
$$= \langle \max(a, \min(1-a, c)), \min(1-a, 1-c)\rangle$$
$$= \Box\langle a, 1-a\rangle \vee_Z \langle c, 1-c\rangle$$
$$= V(\Box x \vee_Z \Box y).$$

3 Conclusion

In a next research we will study the relations between Zadeh's implications, conjunctions and disjunctions from one side, and the other intuitionistic fuzzy operations and operators from another.

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