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Some properties of the averaging operator on intuitionistic fuzzy sets

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Abstract

The averaging operator W is defined on the Intuitionistic Fuzzy Sets (IFSs). The relations between some IFS, on which the averaging operator is applied, are discussed.

Let E be a fixed universe. The averaging operator W [2] is defined on the Intuitionistic Fuzzy Set (IFS, see, e.g., [1])

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in E\}$$

as:

$$\mathrm{W}(A) = \left\{< x, \frac{\sum_{y \in E} \mu_A(y)}{\mathrm{card}(E)}, \frac{\sum_{y \in E} \nu_A(y)}{\mathrm{card}(E)} > | x \in E\right\}$$

if E is discrete and

$$\mathrm{W}(A) = \left\{< x, \frac{\int_E \mu_A(y)\mathrm{d}y}{\|E\|}, \frac{\int_E \nu_A(y)\mathrm{d}y}{\|E\|} > | x \in E\right\}$$

if E is continuous. Here, the relations between some IFS, on which the averaging operator W is applied, are discussed.

Theorem 1 *For every two IFSs A and B the following relations hold:*

$$\mathrm{W}(A \cap B) \subset \mathrm{W}(A) \cap \mathrm{W}(B) \tag{1}$$

$$\mathrm{W}(A \cup B) \supset \mathrm{W}(A) \cup \mathrm{W}(B) \tag{2}$$

$$\mathrm{W}(A@B) = \mathrm{W}(A)@\mathrm{W}(B) \tag{3}$$

$$\mathrm{W}(A\$B) \ \subset_{\Box} \ \mathrm{W}(A)\$\mathrm{W}(B) \qquad \mathrm{W}(A\$B) \ {}_{\Diamond}\!\supset \ \mathrm{W}(A)\$\mathrm{W}(B) \tag{4}$$

$$\mathrm{W}(A \bowtie B) \ \subset_{\Box} \ \mathrm{W}(A) \bowtie \mathrm{W}(B) \qquad \mathrm{W}(A \bowtie B) \ {}_{\Diamond}\!\supset \ \mathrm{W}(A) \bowtie \mathrm{W}(B) \tag{5}$$

$$\mathrm{W}(A - B) \subset \mathrm{W}(A) - \mathrm{W}(B) \tag{6}$$

$$\mathrm{W}(A \to B) \supset \mathrm{W}(A) \to \mathrm{W}(B) \tag{7}$$

and none of the inclusion relations $\subset$, $\subset_{\Box}$ and $\subset_{\Diamond}$ applies to $\mathrm{W}(A+B)$ and $\mathrm{W}(A)+\mathrm{W}(B)$, $\mathrm{W}(A \cdot B)$ and $\mathrm{W}(A) \cdot \mathrm{W}(B)$ and $\mathrm{W}(A * B)$ and $\mathrm{W}(A) * \mathrm{W}(B)$, respectively.

Proof: We consider in details only the case when E is discrete. The proof, when E is continuous, is analogous. Let

$$L_1 = \mathrm{W}(A \cap B), R_1 = \mathrm{W}(A) \cap \mathrm{W}(B),$$

$$L_2 = \mathrm{W}(A \cup B), R_2 = \mathrm{W}(A) \cup \mathrm{W}(B),$$

$$L_3 = \mathrm{W}(A@B), R_3 = \mathrm{W}(A)@\mathrm{W}(B),$$

$$L_4 = \mathrm{W}(A\ B), R_4 = \mathrm{W}(A)\ \mathrm{W}(B),$$

$$L_5 = \mathrm{W}(A - B), R_5 = \mathrm{W}(A) - \mathrm{W}(B),$$

$$L_6 = \mathrm{W}(A \to B), R_6 = \mathrm{W}(A) \to \mathrm{W}(B),$$

$$L_7 = \mathrm{W}(A + B), R_7 = \mathrm{W}(A) + \mathrm{W}(B),$$

$$L_8 = \mathrm{W}(A \cdot B), R_8 = \mathrm{W}(A) \cdot \mathrm{W}(B),$$

$$L_9 = \mathrm{W}(A * B), R_9 = \mathrm{W}(A) * \mathrm{W}(B),$$

$$L_{10} = \mathrm{W}(A \bowtie B), R_{10} = \mathrm{W}(A) \bowtie \mathrm{W}(B).$$

For (1) we have

$$\mu_{L_1}(x) = \frac{1}{\mathrm{card}(E)} \sum_{y \in E} \min(\mu_A(y), \mu_B(y))$$

$$\nu_{L_1}(x) = \frac{1}{\mathrm{card}(E)} \sum_{y \in E} \max(\nu_A(y), \nu_B(y))$$

$$\mu_{R_1}(x) = \frac{1}{\mathrm{card}(E)} \min(\sum_{y \in E} \mu_A(y), \sum_{y \in E} \mu_B(y))$$

$$\nu_{R_1}(x) = \frac{1}{\mathrm{card}(E)} \max(\sum_{y \in E} \nu_A(y), \sum_{y \in E} \nu_B(y))$$

Since

$$\min(\mu_A(y), \mu_B(y)) \leq \mu_A(y) \quad \text{and} \quad \min(\mu_A(y), \mu_B(y)) \leq \mu_B(y) \quad \forall y \in E$$

and

$$\max(\nu_A(y), \nu_B(y)) \geq \nu_A(y) \quad \text{and} \quad \max(\nu_A(y), \nu_B(y)) \geq \nu_B(y) \quad \forall y \in E$$

when we sum for all y in E

$$\sum_{y \in E} \min(\mu_A(y), \mu_B(y)) \leq \sum_{y \in E} \mu_A(y)$$

$$\sum_{y \in E} \min(\mu_A(y), \mu_B(y)) \leq \sum_{y \in E} \mu_B(y)$$

$$\sum_{y \in E} \max(\nu_A(y), \nu_B(y)) \geq \sum_{y \in E} \nu_A(y)$$

$$\sum_{y\in E}\max(\nu_A(y),\nu_B(y)) \geq \sum_{y\in E}\nu_B(y)$$

That is so $\mu_{L_1}(x) \leq \mu_{R_1}(x)$ and $\nu_{L_1}(x) \geq \nu_{R_1}(x)$, which means $L_1 \subset R_1$. For (2) we have

$$\mu_{L_2}(x) = \frac{1}{\text{card}(E)}\sum_{y\in E}\max(\mu_A(y),\mu_B(y))$$

$$\nu_{L_2}(x) = \frac{1}{\text{card}(E)}\sum_{y\in E}\min(\nu_A(y),\nu_B(y))$$

$$\mu_{R_2}(x) = \frac{1}{\text{card}(E)}\max(\sum_{y\in E}\mu_A(y),\sum_{y\in E}\mu_B(y))$$

$$\nu_{R_2}(x) = \frac{1}{\text{card}(E)}\min(\sum_{y\in E}\nu_A(y),\sum_{y\in E}\nu_B(y))$$

The same way we get

$$\sum_{y\in E}\max(\mu_A(y),\mu_B(y)) \geq \sum_{y\in E}\mu_A(y)$$

$$\sum_{y\in E}\max(\mu_A(y),\mu_B(y)) \geq \sum_{y\in E}\mu_B(y)$$

$$\sum_{y\in E}\min(\nu_A(y),\nu_B(y)) \leq \sum_{y\in E}\nu_A(y)$$

$$\sum_{y\in E}\min(\nu_A(y),\nu_B(y)) \leq \sum_{y\in E}\nu_B(y)$$

so $\mu_{L_2}(x) \geq \mu_{R_2}(x)$ and $\nu_{L_2}(x) \leq \nu_{R_2}(x)$ therefore $L_2 \supset R_2$. For (6) and (7) the same way we get

$$\sum_{y\in E}\min(\mu_A(y),\nu_B(y)) \leq \sum_{y\in E}\mu_A(y)$$

$$\sum_{y\in E}\min(\mu_A(y),\nu_B(y)) \leq \sum_{y\in E}\nu_B(y)$$

$$\sum_{y\in E}\max(\mu_B(y),\nu_A(y)) \geq \sum_{y\in E}\nu_A(y)$$

$$\sum_{y\in E}\max(\mu_B(y),\nu_A(y)) \geq \sum_{y\in E}\mu_B(y)$$

which means $\mu_{L_5} \leq \mu_{R_5}$ and $\nu_{L_5} \geq \nu_{R_5}$ so $L_5 \subset R_5$. Also $\mu_{L_6} \geq \mu_{R_6}$ and $\nu_{L_6} \leq \nu_{R_6}$ so $L_6 \supset R_6$. We proved (1), (2), (6) and (7) - these are the cases when we have inequalities in minimums and maximums. In the rest of the cases we have to establish inequalities of type

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i,y_i) \geq f(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{i=1}^{n} y_i) \tag{8}$$

or

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i, y_i) \leq f(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{i=1}^{n} y_i) \tag{9}$$

which are Jensen inequalities. If $f(x, y)$ is convex (8) holds and if $f(x, y)$ is concave (9) holds. A special case is when $f(x, y)$ is linear - then it is both convex and concave and the inequalities turn into equalities. Exactly this is the case with (3)

$$\mu_{L_3}(x) = \frac{1}{\text{card}(E)}\sum_{y\in E}\frac{\mu_A(y) + \mu_B(y)}{2} =$$

$$= \frac{1}{2}\left[\frac{1}{\text{card}(E)}\sum_{y\in E}\mu_A(y) + \frac{1}{\text{card}(E)}\sum_{y\in E}\mu_B(y)\right] = \mu_{R_3}(x)$$

$$\nu_{L_3}(x) = \frac{1}{\text{card}(E)}\sum_{y\in E}\frac{\nu_A(y) + \nu_B(y)}{2} =$$

$$= \frac{1}{2}\left[\frac{1}{\text{card}(E)}\sum_{y\in E}\nu_A(y) + \frac{1}{\text{card}(E)}\sum_{y\in E}\nu_B(y)\right] = \nu_{R_3}(x)$$

so $\text{W}(A@B) = \text{W}(A)@\text{W}(B)$. If $f(x, y)$ is not linear but it has continuous first and second order derivatives - from Taylor's formula we have

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \frac{1}{1!}\left[f'_x(x_0, y_0)h + f'_y(x_0, y_0)k\right] +$$

$$+\frac{1}{2!}\left[f''_{x^2}(x_0 + \theta h, y_0 + \theta k)h^2 + 2f''_{xy}(x_0 + \theta h, y_0 + \theta k)hk + f''_{y^2}(x_0 + \theta h, y_0 + \theta k)k^2\right]$$

for some $\theta \in [0; 1]$. So (8) (or (9)) will hold if and only if

$$f''_{x^2}(x_0 + \theta h, y_0 + \theta k)h^2 + 2f''_{xy}(x_0 + \theta h, y_0 + \theta k)hk + f''_{y^2}(x_0 + \theta h, y_0 + \theta k)k^2 \geq 0 \quad (\text{or} \leq 0$$

for all x_0, y_0 and $x_0 + h, y_0 + k$ for which we consider (8) (or (9)). Therefore (8) (or (9)) will hold if and only if

$$f''_{xy}{}^2(x, y) - f''_{x^2}(x, y)f''_{y^2}(x, y) \leq 0 \tag{10}$$

Now we just have to check (10) for the operations \$, $+$, $\cdot$, $*$ and $\bowtie$. For \$

$$f(x, y) = \sqrt{xy} \qquad f'_x = \frac{1}{2}\sqrt{\frac{y}{x}} \qquad f'_y = \frac{1}{2}\sqrt{\frac{x}{y}}$$

$$f''_{x^2} = -\frac{\sqrt{xy}}{4x^2} \qquad f''_{xy} = \frac{1}{4\sqrt{xy}} \qquad f''_{y^2} = -\frac{\sqrt{xy}}{4y^2}$$

$$f''_{xy}{}^2(x, y) - f''_{x^2}(x, y)f''_{y^2}(x, y) = \frac{1}{16xy} - \frac{1}{16xy} = 0$$

and $f''_{x^2}(x,y) \leq 0$ so $f(x,y)$ is concave and (9) holds. Therefore

$$\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \sqrt{\mu_A(y)\mu_B(y)} \leq \sqrt{\left(\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \mu_A(y)\right)\left(\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \mu_B(y)\right)}$$

$$\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \sqrt{\nu_A(y)\nu_B(y)} \leq \sqrt{\left(\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \nu_A(y)\right)\left(\frac{1}{\mathrm{card}(E)} \sum_{y \in E} \nu_B(y)\right)}$$

which is $\mu_{L_4}(x) \leq \mu_{R_4}(x)$ and $\nu_{L_4}(x) \leq \nu_{R_4}(x)$ so $L_4 \subset_{\square} R_4$ and $L_4 \diamondsuit\!\supset R_4$. Analogically for $\bowtie$

$$f(x,y) = \frac{2xy}{x+y} \qquad f'_x = \frac{2y^2}{(x+y)^2} \qquad f'_y = \frac{2x^2}{(x+y)^2}$$

$$f''_{x^2} = -\frac{4y^2}{(x+y)^3} \qquad f''_{xy} = -\frac{4xy}{(x+y)^3} \qquad f''_{y^2} = -\frac{4x^2}{(x+y)^3}$$

$$f''^{\,2}_{xy}(x,y) - f''_{x^2}(x,y)f''_{y^2}(x,y) = \frac{16x^2y^2}{(x+y)^6} - \frac{16x^2y^2}{(x+y)^6} = 0$$

so $\mu_{L_5}(x) \leq \mu_{R_5}(x)$ and $\nu_{L_5}(x) \leq \nu_{R_5}(x)$ therefore $L_5 \subset_{\square} R_5$ and $L_5 \diamondsuit\!\supset R_5$. For $+$

$$f(x,y) = x + y - xy \qquad f'_x = 1 - y \qquad f'_y = 1 - x$$

$$f''_{x^2} = 0 \qquad f''_{xy} = -1 \qquad f''_{y^2} = 0$$

$$f''^{\,2}_{xy}(x,y) - f''_{x^2}(x,y)f''_{y^2}(x,y) = (-1)^2 - 0 = 1 > 0$$

$$g(x,y) = xy \qquad g'_x = y \qquad g'_y = x$$

$$g''_{x^2} = 0 \qquad g''_{xy} = 1 \qquad g''_{y^2} = 0$$

$$g''^{\,2}_{xy}(x,y) - g''_{x^2}(x,y)g''_{y^2}(x,y) = 1^2 - 0 = 1 > 0$$

so nor (8), nor (9) holds. The same applies to $\cdot$ since the functions of the operation are the same. For $*$

$$f(x,y) = \frac{x+y}{2(xy+1)} \qquad f'_x = \frac{1-y^2}{2(xy+1)^2} \qquad f'_y = \frac{1-x^2}{2(xy+1)^2}$$

$$f''_{x^2} = -\frac{y^3-y}{(xy+1)^3} \qquad f''_{xy} = -\frac{x+y}{(xy+1)^3} \qquad f''_{y^2} = -\frac{x^3-x}{(xy+1)^3}$$

and

$$f''^{\,2}_{xy}(x,y) - f''_{x^2}(x,y)f''_{y^2}(x,y) = \frac{(x+y)^2}{(xy+1)^6} - \frac{(y^3-y)(x^3-x)}{(xy+1)^6}$$

which is evidently positive for some values of x and y in $[0;1]$. (For example if $x = 1$ and $y = 1$ it is $\frac{1}{16}$.) The proof of the theorem is complete.

Another proof of the proposition none of the inclusion relations $\subset$, $\subset_{\Box}$ and $\subset_{\Diamond}$ applies to $\mathrm{W}(A+B)$ and $\mathrm{W}(A)+\mathrm{W}(B)$, $\mathrm{W}(A\cdot B)$ and $\mathrm{W}(A)\cdot\mathrm{W}(B)$ and $\mathrm{W}(A*B)$ and $\mathrm{W}(A)*\mathrm{W}(B)$ respectively, are the following examples: Let $E = \{e_1, e_2, e_3\}$ and

$$A = \{< e_1, \frac{1}{12}, \frac{1}{12} >, < e_2, \frac{3}{12}, \frac{3}{12} >, < e_3, \frac{5}{12}, \frac{5}{12} >\}$$

$$B = \{< e_1, \frac{5}{12}, \frac{1}{12} >, < e_2, \frac{3}{12}, \frac{3}{12} >, < e_3, \frac{1}{12}, \frac{5}{12} >\}$$

$$C = \{< e_1, \frac{1}{12}, \frac{6}{12} >, < e_2, \frac{2}{12}, \frac{2}{12} >, < e_3, \frac{6}{12}, \frac{1}{12} >\}$$

then

$$\mathrm{W}(A+B) \underset{\Box}{\supset} \mathrm{W}(A) + \mathrm{W}(B)$$

$$\mathrm{W}(A+B) \subset_{\Diamond} \mathrm{W}(A) + \mathrm{W}(B)$$

$$\mathrm{W}(A\cdot B) \subset_{\Box} \mathrm{W}(A)\cdot\mathrm{W}(B)$$

$$\mathrm{W}(A\cdot B) \underset{\Diamond}{\supset} \mathrm{W}(A)\cdot\mathrm{W}(B)$$

$$\mathrm{W}(A*B) \subset \mathrm{W}(A)*\mathrm{W}(B)$$

but

$$\mathrm{W}(A+C) \subset_{\Box} \mathrm{W}(A) + \mathrm{W}(C)$$

$$\mathrm{W}(A+C) \underset{\Diamond}{\supset} \mathrm{W}(A) + \mathrm{W}(C)$$

$$\mathrm{W}(A\cdot C) \underset{\Box}{\supset} \mathrm{W}(A)\cdot\mathrm{W}(C)$$

$$\mathrm{W}(A\cdot C) \subset_{\Diamond} \mathrm{W}(A)\cdot\mathrm{W}(C)$$

$$\mathrm{W}(A*C) \supset \mathrm{W}(A)*\mathrm{W}(C).$$

References

[1] Atanassov, K. Intuitionistic Fuzzy Sets. Springer-Verlag, Heidelberg, 1999.

[2] Atanassov K. and A. Ban, On an operator over intuitionistic fuzzy sets. Comptes Rendus de l'Academie bulgare des Sciences, Tome 53, 2000, No. 5, 39-42.