Notes on Intuitionistic Fuzzy Sets
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Some remarks on n-uninorms in IF-sets

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Abstract: Uninorms and nullnorms are well-known monoidal and monotone operations on the unit interval. Akella [2007] proposed their genaralization to n-uninorms. Really, we get both, proper uninorms as well as proper nullnorms as special cases of 2-uninorms. Moreover, proper uninorms as well as proper nullnorms can be characterized as 2-uninorms with some special types of 2-neutral elements. In the present paper, we discuss a classification of 2-uninorms from another point of view as it was done by Akella in 2007 and 2009. Then, we look at 2-uninorms in IF-sets and point out some differences between 2-uninorms on the unit interval and 2-uninorms in IF-sets.

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1 Introduction

Uninorms were introduced by Yager and Rybalov [21] as a generalization of both t-norms and t-conorms (for details on t-norms and their duals, t-conorms, see, e.g., [13, 17]). Since that time, researchers study properties of several distinguished families of uninorms. In [16], Karaçal and Mesiar introduced uninorms in bounded lattices. In [5], Bodjanova and Kalina constructed uninorms in bounded lattices with arbitrarily given underlying t-norm and t-conorm.

Another generalization of t-norms and t-conorms, called t-operators, was introduced by Mas et al. In [18, 19], Mas et al. studied t-operators on finite chains. In 2001, Calvo et al. [6] introduced nullnorms when trying to solve Frank's functional equation [11] where one of the operations in the equation was a uninorm. Afterwards, Mas et al. [20] showed that nullnorms and t-operators coincide in the unit interval. Karaçal et al. [15] introduced nullnorms in bounded lattices.

Akella [1, 2] introduced 2- and n-uninorms in the unit interval and gave a characterization of these operations. In this paper, we characterize 2-uninorms (or more general on n-uninorms) from the point of view of their two-neutral elements. Particularly, we split the system of all 2-uninorms into 9 (not necessarily disjoint) subclasses. Afterwards, we point out some differences in the structure of 2- (and n-) uninorms in IF sets.

Intuitionistic fuzzy sets (also, IF-sets), introduced by Atanassov, are a special type of lattice-valued fuzzy sets, introduced by Goguen [12]. Important milestones in the theory of IF-sets, besides the monograph by Atanassov [3], are the papers by Deschrijver [7, 8], and Deschrijver and Kerre [9]. In [7] Deschrijver has shown that there exist t-norms which are not representable as a pair of a t-norm and a t-conorm. In [8] the author has shown that there exist uninorms in IF-sets which are neither conjunctive nor disjunctive. In [9], Deschrijver and Kerre have shown that the theory of IF-sets is equivalent to the theory of interval-valued sets.

A further development of uninorms in IF-sets (or, equivalently, in interval-valued sets) is the paper by Kalina and Král' [14], where the authors have shown that for arbitrary pair $(\mathbf{a}, \mathbf{e})$ of incomparable elements of interval-valued sets there exists a uninorm having $\mathbf{a}$ as the annihilator and $\mathbf{e}$ as the neutral element.

2 Basic definitions and some known facts

An IF-set [3] can be represented as a special case of L-fuzzy set [12], where L is a bounded lattice. Membership grades of an IF-set are elements $(x_1, x_2) \in [0,1]^2$ such that $x_1 + x_2 \leq 1$. The set of all IF-membership grades will be denoted by L^*. For arbitrary $(x_1, x_2), (y_1, y_2) \in L^*$ the following holds

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \ \Leftrightarrow \ x_1 \leq x_2 \ \& \ y_1 \geq y_2.$$

Thus, the least and the greatest elements of L^* are $\mathbf{0} = (0, 1)$, $\mathbf{1} = (1, 0)$, respectively. We will write these values in bold letters to distinguish them from the real numbers 0 and 1.

Following the notation introduced in [4], we will write $x \parallel y$ if $x, y \in L^*$ are incomparable. For $x \in L^*$ we denote $\parallel_x = \{z \in L^*; z \parallel x\}$.

Definition 1 ([21])**.** *An associative, commutative and monotone operation $U : [0,1]^2 \to [0,1]$ is said to be a uninorm if it has a neutral element $e \in [0,1]$*

A uninorm U has an annihilator $a = U(0,1)$, where $a \in \{0,1\}$.

Definition 2 ([10])**.** *A uninorm U is said to be conjunctive if $U(0,1) = 0$, and U is called disjunctive if $U(0,1) = 1$.*

Lemma 1 ([10])**.** *A uninorm U is a t-norm whenever its neutral element is $e = 1$. In that case the annihilator of U is $a = 0$.*

U is a t-conorm whenever its neutral element is $e = 0$. In that case the annihilator of U is $a = 1$.

Lemma 2 ([10])**.** *Let U be a uninorm, $e \in]0,1[$ be its neutral element. Then*

$$T_U(x,y) = \frac{U(ex,ey)}{e}, \quad S_U(x,y) = \frac{U(e+(1-e)x, e+(1-e)y) - e}{1-e},$$

are a t-norm and a t-conorm, respectively.

The operations T_U and S_U from Lemma 2 are called *the underlying t-norm* and the *underlying t-conorm*, respectively.

Definition 3 ([6])**.** *An associative, commutative and monotone operation $V : [0,1]^2 \to [0,1]$ is said to be a nullnorm if there exists an element $a \in [0,1]$ such that*

(1b) $V(0,x) = x$ *for all* $x \in [0,a]$,

(2b) $V(1,x) = x$ *for all* $x \in [a,1]$.

Lemma 3 ([6])**.** *Let V be a nullnorm and $a \in [0,1]$ be such that*

(1b) $V(0,x) = x$ *for all* $x \in [0,a]$,

(2b) $V(1,x) = x$ *for all* $x \in [a,1]$.

Then a is the annihilator of V.

Similarly like for uninorm U, also for nullnorm V there exist its undrlying t-norm V_T and t-conorm S_T given by, respectively,

$$V_T(x,y) = \frac{V(a+(1-a)x, a+(1-a)y) - a}{1-a}, \quad V_S(x,y) = \frac{V(ax,ay)}{a}$$

for $a \in]0,1[$.

Definition 4 ([1])**.** *Let $F : [0,1]^2 \to [0,1]$ be a commutative operation. Then $\{e_1, e_2\}_z$ is called a 2-neutral element of F if $F(e_1, x) = x$ for all $x \in [0,z]$ and $V(e_2, x) = x$ for all $x \in [z,1]$, where $0 < z < 1$ and $e_1 \in [0,z]$, $e_2 \in [z,1]$.*

Definition 5 ([1])**.** *Let $F : [0,1]^2 \to [0,1]$ be a monotone, commutative and associative operation that has a 2-neutral element $\{e_1, e_2\}_z$.*

Lemma 4 ([1])**.** *Let F be a 2-uninorm whose 2-neutral element is $\{e_1, e_2\}_z$. Then*

$$U_1(x,y) = \frac{F(zx,zy)}{z}, \quad U_2(x,y) = \frac{F(z+(1-z)x, z+(1-z)y) - z}{1-z} \tag{1}$$

are uninorms whose neutral elements are $\tilde{e}_1 = \frac{e_1}{z}$ and $\tilde{e}_2 = \frac{e_2 - z}{1-z}$, respectively.

Definition 6 ([1])**.** *Let $F : [0,1]^2 \to [0,1]$ be a commutative operation and $0 = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n = 1$. Then $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})}$ is called an n-neutral element of F if for all $i \in \{1, 2, \ldots, n\}$ we have $e_i \in [z_{i-1}, z_i]$.*

Definition 7 ([1])**.** *An associative, commutative and monotone operation $F : [0,1]^2 \to [0,1]$ will be called n-uninorm if it has an n-neutral element $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})}$.*

3 Characterization and classes of 2-uninorms and a generalization to n-uninorms

Let us consider proper uninorms and proper nullnorms as 1-uninorms. Then we adopt the the following definition:

Definition 8. *Let F_n be an n-uninorm for $n > 1$. We say that F_n is a proper n-uninorm if F_n is not an $(n-1)$-uninorm.*

For a proper 2-uninorm F, the operations U_1 and U_2 given by equality (1), will be called *the lower and the upper underlying uninorm*, respectively.

Let us a look at 2-neutral elements. For a given $0 < z < 1$, there are 9 possibilities how to set a 2-neutral element $\{e_1, e_2\}_z$. Namely,

$$e_1 \begin{cases} = 0, \\ \in \left]0, z\right[, \\ z, \end{cases} \qquad e_2 \begin{cases} = z, \\ \in \left]z, 1\right[, \\ 1. \end{cases}$$

As a corollary to Lemma 1 we get the following

Corollary 1. *Let F be a 2-uninorm whose 2-neutral element is $\{e_1, e_2\}_z$ for $z \in \left]0,1\right[$. Set $U_1(x,y) = \frac{F(zx,zy)}{z}$ and $U_2(x,y) = \frac{F(z+(1-z)x, z+(1-z)y)-z}{1-z}$ Then*

(a) U_1 is a t-norm if $e_1 = z$, U_2 is a t-norm if $e_2 = 1$;

(b) U_1 is a proper uninorm if $e_1 \in \left]0, z\right[$, U_2 is a proper uninorm if $e_2 \in \left]z, 1\right[$;

(c) U_1 is a t-conorm if $e_1 = 0$, U_2 is a t-conorm if $e_2 = z$.

Let us check all 9 possibilities of setting a 2-neutral element.

Lemma 5. *Let F be a 2-uninorm whose 2-neutral element is $\{z, 1\}_z$ ($\{0, z\}_z$) for $0 < z < 1$. Then F is a t-norm (t-conorm) that is the ordinal sum of two t-norms $F = (\langle T_1, 0, z\rangle, \langle T_2, z, 1\rangle)$ (of two t-conorms $F = (\langle S_1, 0, z\rangle, \langle S_2, z, 1\rangle)$).*

Proof. We will prove only the t-norm case.
As the first step, let us prove that $F(0,1) = 0$. Since $\{z,1\}_z$ is the 2-neutral element of F, we have by associativity

$$F(0,1) = F(F(0,z),1) = F(0, F(z,1)) = F(0,z) = 0.$$

Monotonicity of F implies that 0 is the annihilator of F.
As the second step, we prove that 1 is the neutral element of F. Since we know that 1 is the partial neutral element of F in the interval $[z,1]$. Let $x \in [0,z]$.

$$F(x,1) = F(F(x,z),1) = F(x, F(z,1)) = F(x,z) = x.$$

The proof is completed. $\square$

Lemma 6. *Let F be a 2-uninorm whose 2-neutral element is $\{z\}_z$ ($\{e_1, z\}_z$, $\{z, e_2\}_z$) for $0 < z < 1$ and $0 < e_1 < z$, $z < e_2 < 1$. Then F is a uninorm whose neutral element is z (e_1 and the underlying t-conorm S is the ordinal sum of two t-conorms $S = (\langle S_1, e_1, z\rangle, \langle S_2, z, 1\rangle)$, e_2 and the underlying t-norm T is the ordinal sum of two t-norms $T = (\langle T_1, 0, z\rangle, \langle T_2, z, e_2\rangle)$).*

Proof. In the case that $\{z\}_z$, we have that z is a partial neutral element in the interval $[0, z]$ as well as in the interval $[z, 1]$, i.e., z is the neutral element of F. Hence, directly by Definition 1 we get that F is a uninorm with the neutral element z.

In the case that $\{e_1, z\}_z$ is the 2-neutral element of F, we get applying Lemma 5 to the interval $[e_1, 1]$ that $S = (\langle S_1, e_1, z\rangle, \langle S_2, z, 1\rangle)$ is a t-conorm which is the underlying operation of F. The rest of the proof is due to Definition 1.

Dually we could prove the case when $\{z, e_2\}_z$ is the 2-neutral element of F. □

Lemma 7. *Let F be a 2-uninorm whose 2-neutral element is $\{0, 1\}_z$ for $0 < z < 1$. Then, F is a nullnorm and z is its annihilator.*

Proof. The fact that F is a nullnorm with the annihilator z is directly due to Definition 3. □

The remaining three cases lead to proper 2-uninorms.

Lemma 8. *Let F be a 2-uninorm whose 2-neutral element is $\{e_1, e_2\}_z$ for $0 < e_1 < z < e_2 < 1$. Then F is a proper 2-uninorm.*

We omit the proof of this lemma since the assertion is obvious.

Lemma 9. *Let F be a 2-uninorm whose 2-neutral element is $\{e, 1\}_z$ for $0 < e < z < 1$. Then F is a proper 2-uninorm whose upper underlying uninorm is reduced to a t-norm.*

Proof. The fact that the upper underlying uninorm is reduced to a t-norm is due to Lemma 5. The rest of the proof is obvious. □

Lemma 10. *Let F be a 2-uninorm whose 2-neutral element is $\{0, e\}_z$ for $0 < z < e < 1$. Then F is a proper 2-uninorm whose lower underlying uninorm is reduced to a t-conorm.*

The assertion of Lemma 10 is a dual case of Lemma 9. That is why the proof is omitted.

Generalizing Lemma 6, we get the following

Proposition 1. *For $n \geq 2$, let F be a proper n-uninorm where $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})}$ is its n-neutral element. Then there exists $1 \leq i \leq n$ such that $z_{i-1} < e_i < z_i$ and moreover, F is an $(n+1)$-uninorm whose $(n+1)$-neutral element is $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, e_i, z_i, \ldots, z_{n-1})}$.*

Proof. We have to prove two items for $n \geq 2$:

1) There exists i such that $z_{i-1} < e_i < z_i$,

2) $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, e_i, z_i, \ldots, z_{n-1})}$ is an $(n+1)$-neutral element of F.

To prove item 1), it is enough to realize that, for $n \geq 2$, if there were no i such that $z_{i-1} < e_i < z_i$, the operation F would have diagonal blocks either $(T_1, S_1, T_2, S_2, \dots)$ or $(S_1, T_1, S_2, T_2, \dots)$, where T_1, T_2 are t-norms, and S_1, S_2 are t-conorms. In each of these two cases the n-neutral element could be reduced to the $(n-1)$-neutral element, since in the first case $e_1 = e_2$ and in the second case $e_2 = e_3$ either $\{e_2, \dots, e_n\}_{(z_2, \dots, e_i, z_i, \dots, z_{n-1})}$ or $\{e_1, \dots, e_n\}_{(z_1, \dots, e_i, z_i, \dots, z_{n-1})}$, respectively. This proves the item 1) for $n \geq 3$. For $n = 2$ the statement is due to Lemmas 8, 9 and 10.

Item 2) is a direct consequence of item 1). □

4 2-uninorms on IF-sets

$(L^*, \leq_{L^*})$ is a bounded lattice with incomparable elements. The incomparability of some elements will be crucial in our considerations.

Example 1. On the bounded lattice $(L^*, \wedge, \vee, \mathbf{0}, \mathbf{1})$, $T_\wedge(z_1, z_2) = z_1 \wedge z_2$ is the greatest t-norm. When we choose an arbitrary element $x \notin \{\mathbf{0}, \mathbf{1}\}$, $T_\wedge$ can be considered as the ordinal sum t-norm $(\langle T_\wedge, \mathbf{0}, x\rangle, \langle T_\wedge, x, \mathbf{1}\rangle)$.

On the other hand, since $\|_x \neq \emptyset$, we can define

$$\tilde{T}_\wedge(z_1, z_2) = \begin{cases} z_1 \wedge z_2 & \text{for } (z_1, z_2) \in ([\mathbf{0}, x] \cup [x, \mathbf{1}])^2, \\ z_2 & \text{for } z_1 \in \|_x,\ z_2 \in [\mathbf{0}, x] \\ z_1 & \text{for } z_1 \in [\mathbf{0}, x],\ z_1 \in \|_x, \\ x & \text{otherwise.} \end{cases} \tag{2}$$

Hence, $\tilde{T}_\wedge$ is not a t-norm, but $\{x, \mathbf{1}\}_x$ is a 2-neutral element of $\tilde{T}_\wedge$. This means that $\tilde{T}_\wedge$ is a proper 2-uninorm.

Example 2. Let $x \notin \{\mathbf{0}, \mathbf{1}\}$ be an element in L^*.

$$\tilde{U}(z_1, z_2) = \begin{cases} z_1 \wedge z_2 & \text{for } (z_1, z_2) \in [\mathbf{0}, x]^2, \\ z_1 & \text{for } z_1 \in [\mathbf{0}, x] \text{ and } z_2 \notin [\mathbf{0}, x], \\ & \text{ and for } z_1 \in [x, \mathbf{1}] \text{ and } z_2 \in \|_x, \\ z_2 & \text{for } z_2 \in [\mathbf{0}, x] \text{ and } z_1 \notin [\mathbf{0}, x], \\ & \text{ and for } z_2 \in [x, \mathbf{1}] \text{ and } z_1 \in \|_x, \\ z_1 \vee z_2 & \text{for } (z_1, z_2) \in [x, \mathbf{1}]^2, \\ x & \text{otherwise.} \end{cases} \tag{3}$$

The operation $\tilde{U}$ is restricted to $[\mathbf{0}, x] \cup [x, \mathbf{1}]$, if $\tilde{U}$ has no neutral element on the whole L^*, i.e., it is not a uninorm. On the other hand, $\{x\}_x$ is a 2-neutral element, hence $\tilde{U}$ is a proper 2-uninorm.

Example 3. Let $x \notin \{\mathbf{0}, \mathbf{1}\}$ be an element in L^*.

$$V(z_1, z_2) = \begin{cases} z_1 \vee z_2 & \text{for } (z_1, z_2) \in [\mathbf{0}, x]^2, \\ z_1 \wedge z_2 & \text{for } (z_1, z_2) \in [x, \mathbf{1}]^2, \\ x & \text{otherwise.} \end{cases} \tag{4}$$

V is a nullnorm whose annihilator is x. In this case, if we are looking for a modification $\tilde{V}$ of V in such a way that $\tilde{V}$ is reduced to $[\mathbf{0}, x] \cup [x, \mathbf{1}]$, but $\tilde{V}$ is not a nullnorm, we will not succeed. Really, we have that $V(\mathbf{1}, \mathbf{0}) = x$ and hence also $\tilde{V}(\mathbf{1}, \mathbf{0}) = x$ and this implies that x is the annihilator of $\tilde{V}$.

Remark 1. Dually to the operation $\tilde{T}_\wedge$ introduced by (2), we can define on L^* an operation $\tilde{S}_\vee$ starting from the t-conorm $S_\vee(z_1, z_2) = z_1 \vee z_2$ and an element $x \notin \{\mathbf{0}, \mathbf{1}\}$. This means that, unlike the situation with the operations in the unit interval, an arbitrary form of the 2-neutral element, except of the case when $\{\mathbf{0}, \mathbf{1}\}_x$ is the 2-neutral element, may lead to proper 2-uninorms.

As a corollary to the above considerations in Examples 1 – 3, we get the following proposition.

Proposition 2. *For arbitrary $n \geq 2$ there exists a proper n-uninorm $F : L^* \times L^* \to L^*$ such that F has no $(n + 1)$-neutral element, i.e., F is not an $(n + 1)$-uninorm.*

Proof. It is enough to modify the construction in Example 1. For arbitrary $n \geq 2$, let us choose $\mathbf{0} = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-1} < \zeta_n = \mathbf{1}$ and we define an operation $\tilde{T}$ by

$$\tilde{T}(z_1, z_2) = \begin{cases} z_1 & \text{for } z_1 \in [\zeta_{i-1}, \zeta_i],\ i \in \{1, 2, \dots n-1\} \text{ and } z_2 \geq \zeta_i, \\ z_2 & \text{for } z_2 \in [\zeta_{i-1}, \zeta_i],\ i \in \{1, 2, \dots n-1\} \text{ and } z_1 \geq \zeta_i, \\ \zeta_{i-1} & \text{for } i \in \{1, 2, \dots, n\} \text{ and } (z_1, z_2) \in [\zeta_{i-1}, \zeta_i[^2, \\ & \text{ or } (z_1, z_2) \in [\zeta_{i-1}, \mathbf{1}]^2 \text{ and } z_1 \parallel_{\zeta_i} \text{ or } z_2 \parallel_{\zeta_i}. \end{cases}$$

We get that $\{\zeta_1, \zeta_2, \dots, \zeta_n\}_{(\zeta_1, \zeta_2, \dots, \zeta_{n-1})}$ is the n-neutral element of $\tilde{T}$ and there exists no $(n + 1)$-neutral element of $\tilde{T}$. □

5 Conclusions

In this paper, we have discussed 2-uninorms in the unit interval and in the L^* lattice of IF-membership grades. We have shown that there are substantial differences between 2-uninorms in the unit interval and 2-uninorms in the L^* lattice. The results on 2-uninorms we have generalized to n-uninorms.

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