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A new similarity measure and new distances for intuitionistic fuzzy sets

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Abstract

This paper is a continuation of our previous works on similarity measures of Atanassov's intuitionistic fuzzy sets (to be called A-IFSs, for short). The similarity measures we considered used all three functions (membership, non-membership and hesitation) to represent A-IFSs, and examined two kinds of distances – one to an object to be compared, and one to its complement. In this paper we propose some new distances between A-IFSs, and new similarity measures preserving all the advantages of the previously proposed similarity measures and using the new distance functions.

1 Introduction

Similarity measures play a fundamental role in inference and approximate reasoning, and in virtually all applications of fuzzy logic. For different purposes different measures of similarity are to be used. The importance of those measures has motivates researchers to compare and examine the effectiveness and properties of different measures of similarity for fuzzy sets (e.g. Zwick at al. [27], Pappis and Karacapilidis [7], Chen at al. [4], Wang at al. [25], Bouchon-Meunier et al. [3], Cross and Sudkamp [5]).

The analysis of similarity is also a fundamental issue while employing A-IFSs (Atanassov [1], [2]) which are a generalization of conventional fuzzy sets.

Like in our previous works (Szmidt and Kacprzyk [17], [18], [21]) we propose here a similarity measure which is not a standard similarity measure in the sense that it is not only a dual concept to a (general) distance measure (cf. Tversky [23]). In commonly used similarity measures, the dissimilarity behaves like a distance function. Such a standard approach, formulated for objects meant as crisp values, was later extended and used to evaluate the similarity of fuzzy sets (Cross and Sudkamp [5]). Distances were also proposed to measure the similarity between intuitionistic fuzzy sets (cf. Dengfeng and Chuntian [6], and Szmidt and Kacprzyk [17], [18], [21]).

The measure we propose here is a different kind of a similarity measure as it does not measure just a distance between individual intuitionistic fuzzy preferences being compared.

The new measure answers the question if the compared preferences are more similar or more dissimilar to each other.

The second feature (besides taking into account the complements of the element/object we compare to) that makes our measure distinct is the representation of A-IFSs as we take into account all three functions i.e., the membership, non-membership, and hesitation margin. We have already discussed the reasons of such an represenation of A-IFSs (e.g., Szmidt and Kacprzyk [11], [13], [20]).

Should similarity measures between A-IFSs be just a straightforward generalization of measures between fuzzy sets? The results obtained show that, just as in the case of distances (Szmidt and Kacprzyk [20]), straightforward approaches may not work (Szmidt and Kacprzyk [21]).

In this paper we propose some new distances between A-IFSs, and new similarity measures preserving all the advantages of the previously proposed similarity measures and using the new distance functions.

The basic definitions and properties of Atanassov's intuitionistic fuzzy sets are given in [1], [2].

2 Distances between A-IFSs

In Szmidt and Kacprzyk [11], Szmidt and Baldwin [8, 9], and especially in Szmidt and Kacprzyk [20] it is shown why when calculating distances between IFSs we should take into account all three functions describing A-IFSs. In [20] not only the reasons why we should take into account all three functions are given but also some possible serious problems that can occur while taking into account two functions only.

Here we will consider the folowing distances between A-IFSs A, B in $X = \{x_{1,}, \ldots, x_n\}$:

$$l_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \tag{1}$$

$$l'^{n}_{IFS}(A,B) = \frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{k^2}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \tag{2}$$

$$l'''^{n}_{IFS}(A,B) = \frac{6}{\pi^2}\sum_{k=1}^{n}\frac{1}{k^2}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)) \tag{3}$$

$$l''^{n}_{IFS}(A,B) = \sum_{k=1}^{n}\frac{1}{2^k}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \tag{4}$$

$$l''''^{n}_{IFS}(A,B) = \sum_{k=1}^{n}\frac{1}{2^k}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)) \tag{5}$$

For (1) we have: $0 \leq l_{IFS}(A,B) \leq 1$ (cf. Szmidt and Kacprzyk [11], [20], Szmidt and Baldwin [8, 9]), the values of the distances (2)–(5) are from interval [0, 1).

In our further considerations we will use distances between fuzzy sets A, B in $X = \{x_i/i \in N\}$

$$l'_{IFS}(A,B) = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \quad (6)$$

$$l'''_{IFS}(A,B) = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)) \quad (7)$$

$$l''_{IFS}(A,B) = \sum_{k=1}^{\infty}\frac{1}{2^k}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \quad (8)$$

$$l''''_{IFS}(A,B) = \sum_{k=1}^{\infty}\frac{1}{2^k}(\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|)) \quad (9)$$

Proposition 1 Q is an A-IFS on $X = \{x_i/i \in N\}$. We will prove that $< Q, l'_{IFS}(A,B) >$ and $< Q, l''_{IFS}(A,B) >$ are metrics spaces.

Proof First we will show that $l'_{IFS}(A,B)$ and $l''_{IFS}(A,B)$. As well as $\frac{6}{\pi^2}\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = 1$ and $\sum\limits_{k=1}^{\infty}\frac{1}{2^k} = 1$.

From $\frac{1}{2k}\sum\limits_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|) \leq 1$ and comparison test we have that $l'_{IFS}(A,B)$ and $l''_{IFS}(A,B)$ diverge. So it is obvious that $l'_{IFS}(A,B)$ and $l''_{IFS}(A,B)$ exist, and both $l'_{IFS}(A,B) \leq 1$ and $l''_{IFS}(A,B) \leq 1$.

1) It is obvious that: $l'_{IFS}(A,B) \geq 0$, and $l''_{IFS}(A,B) \geq 0$, $(\frac{1}{2k}\sum\limits_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)| + |\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) \geq 0)$. $l'_{IFS}(A,B) = 0$ and $l''_{IFS}(A,B) = 0$ iff $A = B$.

$\Leftarrow$ If $A = B$ then $\frac{1}{2k}\sum\limits_{i=1}^{k}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) = 0$. Hence $l'_{IFS}(A,B) = 0$ and $l''_{IFS}(A,B) = 0$.

$\Rightarrow$ Let $A \neq B$ then exists $k = k_0$ such that $\frac{1}{2k_0}\sum\limits_{i=1}^{k_0}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)| + |\pi_A(x_i)-\pi_B(x_i)|)) > 0$. Hence $l'_{IFS}(A,B) > \frac{6}{\pi^2}\frac{1}{k_0^2}(\frac{1}{2k_0}\sum\limits_{i=1}^{k_0}(|\mu_A(x_i)-\mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) > 0$ and $l''_{IFS}(A,B) > \frac{1}{2_0^k}(\frac{1}{2k_0}\sum\limits_{i=1}^{k_0}(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i) - \nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) > 0$.

2) $l'_{IFS}(A,B) = l'_{IFS}(B,A)$ and $l''_{IFS}(A,B) = l''_{IFS}(B,A)$. It is obvious that $(\frac{1}{2k}\sum\limits_{i=1}^{k}(|\mu_A(x_i) - \mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|)) = \frac{1}{2k}\sum\limits_{i=1}^{k}(|\mu_B(x_i)-\mu_A(x_i)|+|\nu_B(x_i)-\nu_A(x_i)| + |\pi_B(x_i)-\pi_A(x_i)|)))$.

3) $l'_{IFS}(A,B) \leq l'_{IFS}(A,C) + l'_{IFS}(C,B)$ and $l''_{IFS}(A,B) \leq l''_{IFS}(A,C) + l''_{IFS}(C,B)$. For every k we have $\frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)) \leq \frac{1}{2k}\sum_{i=1}^{k}(|\mu_A(x_i) - \mu_C(x_i)| + |\nu_A(x_i) - \nu_C(x_i)| + |\pi_A(x_i) - \pi_C(x_i)|)) + \frac{1}{2k}\sum_{i=1}^{k}(|\mu_C(x_i) - \mu_B(x_i)| + |\nu_C(x_i) - \nu_B(x_i)| + |\pi_C(x_i) - \pi_B(x_i)|))$. Hence $l'^{n}_{IFS}(A,B) \leq l'^{n}_{IFS}(A,C) + l'^{n}_{IFS}(C,B)$ and $l''^{n}_{IFS}(A,B) \leq l''^{n}_{IFS}(A,C) + l''^{n}_{IFS}(C,B)$. We have $\lim_{n\to\infty} l'^{n}_{IFS}(A,B) \leq \lim_{n\to\infty} l'^{n}_{IFS}(A,C) + \lim_{n\to\infty} l'^{n}_{IFS}(C,B)$ and $\lim_{n\to\infty} l''^{n}_{IFS}(A,B) \leq \lim_{n\to\infty} l''^{n}_{IFS}(A,C) + \lim_{n\to\infty} l''^{n}_{IFS}(C,B)$ where $\lim_{n\to\infty} l'^{n}_{IFS}(A,B) = l'_{IFS}(A,B)$ and $\lim_{n\to\infty} l''^{n}_{IFS}(A,B) = l''_{IFS}(A,B)$.

3 Similarity measures

In this section we construct the similarity measures between A-IFSs in the sense of Szmidt and Kacprzyk [20], [21], i.e., in the proposed measures we take into account as well the hesitation margins (besides membership and non-membership values). And more so, the measures we propose take into account not only a pure distance between compared elements but also answers the questions if the considered elements/objects are more similar or more dissimilar to each other (the measure takes into account and compares two types of distances – to the element/object, and to its complement). We have already shown (Szmidt and Kacprzyk [18]) that even if a distance between the objects compared is small, it can happen that the objects are completely dissimilar.

In this spirit, when constructing the new similarity measures we use here the same two kinds of distances as in Szmidt and Kacprzyk [20], [21] (i.e., $l_{IFS}(X,F)$, $l_{IFS}(X,F^C)$) but now we look for a function with values from $[0,1]$ (cf. Szmidt and Kacprzyk [21]). Specifically, given by

$$f(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F) + l_{IFS}(X,F^C)} \tag{10}$$

$$f(l'^{n}_{IFS}(X,F), l'^{n}_{IFS}(X,F^C)) = \frac{l'^{n}_{IFS}(X,F)}{l'^{n}_{IFS}(X,F) + l'^{n}_{IFS}(X,F^C)} \tag{11}$$

$$f(l''^{n}_{IFS}(X,F), l''^{n}_{IFS}(X,F^C)) = \frac{l''^{n}_{IFS}(X,F)}{l''^{n}_{IFS}(X,F) + l''^{n}_{IFS}(X,F^C)} \tag{12}$$

The above functions are constructed under the condition that we exclude from our considerations the case when $X = F = F^C$ which is, by obvious reasons, not interesting in practice. The assumption $X = F = F^C$ means that we try to compare an element (represented by) X about which we know nothing, to another element about which we know nothing $F = F^C$ So we exclude from our considerations the cases for which $l_{IFS}(X,F) = l_{IFS}(X,F^C) = 0$. Other cases are presented in Table 1.

In this way we have constructed a function which takes into account the same two distances like the previous measure (cf. Szmidt and Kacprzyk [17], [18], [21]) but now the new measure is normalized (its values are in $[0,1]$). It is obvious (see Table 1) that (10) is

Table 1: Possible values of (10) $c, d \in (0,1)$

$l_{IFS}(X,F)$		$l_{IFS}(X,F^C)$	f
0		1	0
1		0	1
1		1	0.5
c	less than	d	c/(c+d)<0.5
c	bigger than	d	d/(c+d)>0.5
c	equal to	d	0.5

a dual concept to a similarity measure (if (10) is equal to zero then similarity is equal to 1; if (10) is equal to 1 then similarity is equal to zero, and so on). In other words, we may use (10) to construct a similarity measure.

As

$$0 \le f(l_{IFS}(X,F), l_{IFS}(X,F^C)) \le 1 \tag{13}$$

we would like to find such a monotone decreasing function g that:

$$g(1) \le g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) \le g(0) \tag{14}$$

which means that

$$0 \le g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1) \le g(0) - g(1) \tag{15}$$

$$0 \le \frac{g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \le 1 \tag{16}$$

In this way we obtain a function having the properties of a similarity measure in a sense that it is monotone decreasing function of (10).

Definition 1

$$Sim(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{g(f(l_{IFS}(X,F), l_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \tag{17}$$

$$Sim(l'^{n}_{IFS}(X,F), l'^{n}_{IFS}(X,F^C)) = \frac{g(f(l'^{n}_{IFS}(X,F), l'^{n}_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \tag{18}$$

$$Sim(l''^{n}_{IFS}(X,F), l''^{n}_{IFS}(X,F^C)) = \frac{g(f(l''^{n}_{IFS}(X,F), l''^{n}_{IFS}(X,F^C))) - g(1)}{g(0) - g(1)} \tag{19}$$

where $(f(l_{IFS}(X,F), l_{IFS}(X,F^C))$ is given by (10), $(f(l'^{n}_{IFS}(X,F), l'^{n}_{IFS}(X,F^C))$ is given by (11), $(f(l''^{n}_{IFS}(X,F), l''^{n}_{IFS}(X,F^C))$ is given by (12)

The simplest function g which may be applied is

$$g(x) = 1 - x \tag{20}$$

which gives from (17), for two A-IFSs A and B with n elements:

$$Sim(l_{IFS}(A,B), l_{IFS}(A,B^C)) = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{l_{IFS}((A(x_i), B(x_i))}{l_{IFS}((A(x_i), B(x_i)) + l_{IFS}((A(x_i), B(x_i)^C)} \tag{21}$$

4 Conclusions

We have proposed the new similarity measures for A-IFSs. The new measures are an effect of our previous considerations on possible representations of A-IFSs (Szmidt and Kacprzyk [11], Tasseva at al. [22]) and distances between A-IFSs (Szmidt and Kacprzyk [11], [20])).

The new similarity measures take into account all three functions (membership, non-membership and hesitation) in the description of A-IFSs, and take into account the complement of the element/object we compare to.

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