On the weak intuitionistic fuzzy implication

based on △ operation

Lilija Atanassova[1] and Piotr Dworniczak[2]

[1] Institute of Information and Communication Technologies, Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2, Sofia-1113, Bulgaria
e-mail: `l.c.atanassova@gmail.com`

[2] The Great Poland University of Social and Economics
ul. Surzyńskich 2, 63-000 Środa Wlkp., Poland
e-mail: `p.dworniczak@wwsse.pl`

Abstract: In this paper, a new weak intuitionistic fuzzy implication is introduced. Fulfillment of some axioms and properties, together with the Modus Ponens and Modus Tollens inference rules are investigated. Negation induced by the newly proposed implication is presented.
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1   Introduction

In 1983, Atanassov presented in [1] a concept of a kind of vague sets, that were named Intuitionistic Fuzzy Sets (IFSs). The concept was directly inspired by the concept of Fuzzy Sets (FSs) introduced by Zadeh in 1965. IFS, however, differs from FS in the two independently defined functions that assign the membership degree and the non-membership degree of a given element x to a given set A. While in FSs the degree of non-membership of the element x to the FS A is equal to $1 - \mu_A(x)$, where $\mu_A(x)$ is the membership degree, Atanassov introduced separate values $\mu_A(x)$ and $\nu_A(x)$ of membership and non-membership of x to the IFS A.

In the so developed Intuitionistic Fuzzy Logic (IFL) a variable x has its truth-value presented by the ordered pair $\langle a, b \rangle$, for which it holds that a, b, $a + b \in [0, 1]$. Such a pair is called an Intuitionistic Fuzzy Pair (IFP), presented in details in [5]. The numbers a and b are interpreted as the degrees of validity and non-validity of x, respectively. We denote the truth-value of x by $V(x)$.

The variable with truth-value *Truth*, as in the classical logic, will be denoted by $\underline{1}$ and the variable *Falsity* will be denoted by $\underline{0}$. Therefore, for these variables it holds that $V(\underline{1}) = \langle 1, 0 \rangle$ and $V(\underline{0}) = \langle 0, 1 \rangle$. In addition, the variable having the truth-value $\langle 0, 0 \rangle$ is also used, symbolically presented as $V(\underline{FI}) = \langle 0, 0 \rangle$. It is called in the literature Full Ignorance ($\underline{FI}$) or Uncertainty. Notably, such a variable does not exist in either the classical or the fuzzy logic.

We call the variable x an Intuitionistic Fuzzy Tautology (IFT), if and only if (shortly: <u>iff</u>) it holds for $V(x) = \langle a, b \rangle$ that $a \geq b$ and, similarly, we call x an Intuitionistic Fuzzy co-Tautology (IFcT), if it holds that $a \leq b$. For every x we can define the value of negation of x in the typical form $V(\neg x) = \langle b, a \rangle$.

For the IF pairs different operations can be defined. One of them is introduced in [12] and later considered in [13].

Definition 1. ([12, p. 24]) *For two IFPs $\langle a, b \rangle$ and $\langle c, d \rangle$, where $a + b + c + d > 0$, the operation $\triangle$ is defined as follows:*

$$\langle a, b \rangle \triangle \langle c, d \rangle = \left\langle \frac{a+c}{a+b+c+d}, \frac{b+d}{a+b+c+d} \right\rangle.$$

Additionally, we assume that

$$\langle 0, 0 \rangle \triangle \langle 0, 0 \rangle = \langle 0.5, 0.5 \rangle.$$

For the following considerations we introduce first some ordering relation for the intuitionistic truth-values. For $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$, where a, b, c, d, $a + b$, $c + d \in [0, 1]$, we denote $V(x) \preceq V(y)$, <u>iff</u> $a \leq c$ and $b \geq d$. Here, the notation $V(x) \succeq V(y)$ means $V(y) \preceq V(x)$.

One of the important logical connectives in the IFL is the Intuitionistic Fuzzy Implication (IFI). In this paper, we will omit the formal difference between an *implication* as a logical connective and an *implicator* as a binary operator, although for some considerations, this difference can be important.

The general conditions for the IFI were given first by Cornelis and Deschrijver [15], Cornelis, Deschrijver and Kerre [17, 18], Cornelis, Deschrijver, Cock and Kerre [16], and later by Liu and Wang [24], and Zhou, Wu and Zhang [26]. These conditions are grounded on the conditions formulated for the classical fuzzy implication (see e.g. [14], Def. 1.1.1., p. 2).

Definition 2. ([15, Def. 4.2, p.6]) *Let $V(x)$, $V(x_1)$, $V(x_2)$, $V(y)$, $V(y_1)$, $V(y_2) \in L$ be any IF truth-values (IFPs). The intuitionistic fuzzy implication is the mapping $I : L^2 \to L$, fulfilling the properties:*

(IFI 1) if $V(x_1) \preceq V(x_2)$, then $I(V(x_1), V(y)) \succeq I(V(x_2), V(y))$,

(IFI 2) if $V(y_1) \preceq V(y_2)$, then $I(V(x), V(y_1)) \preceq I(V(x), V(y_2))$,

(IFI 3) $I(V(\underline{1}), V(\underline{1})) = V(\underline{1})$,

(IFI 4) $I(V(\underline{1}), V(\underline{0})) = V(\underline{0})$,

(IFI 5) $I(V(\underline{0}), V(\underline{0})) = V(\underline{1})$,

(IFI 6) $I(V(\underline{0}), V(\underline{1})) = V(\underline{1})$.

We can see that the condition (IFI 6) can be omitted. The (IFI 6) condition can be obtained as a corollary from the (IFI 5) and (IFI 2) conditions.

In the existing literature, there is the definition of the intuitionistic fuzzy implicator (implication) without the conditions (IFI 1) and (IFI 2) (see e.g. [25, Def. 10, p. 3]). It is, however, an isolated case, and, moreover, neglecting the monotonicity conditions (IFI 1) and (IFI 2), it is inappropriate as it allows too much freedom in defining the 'implicator' or 'implication'.

In the literature on the subject, almost 200 different intuitionistic fuzzy implications have been introduced (see e.g. [2-4]). One of them is presented by Atanassova in [6]. Such kind of implication is called by Dworniczak in [19] a Weak Intuitionistic Fuzzy Implication (WIFI). The WIFIs are studied in [7–11, 20–22].

Definition 3. ([19, Def. 2, p. 13]). *The Weak Intuitionistic Fuzzy Implication (WIFI) is the logical connective $\Rightarrow$ fulfilling the conditions:*

(WIFI 1)   *if $V(x_1) \preceq V(x_2)$, then $V(x_1 \Rightarrow y) \succeq V(x_2 \Rightarrow y)$,*

(WIFI 2)   *if $V(y_1) \preceq V(y_2)$, then $V(x \Rightarrow y_1) \preceq V(x \Rightarrow y_2)$,*

(WIFI 3) $\underline{0} \Rightarrow y$ *is an IFT,*

(WIFI 4) $x \Rightarrow \underline{1}$ *is an IFT,*

(WIFI 5) $\underline{1} \Rightarrow \underline{0}$ *is an IFcT.*

We call the operation 'weak' because the (WIFI 3)–(WIFI 5) conditions have been mainly defined in the 'strong' form as (IFI 3)–(IFI 6) (see e.g. [15, 17]).

The most important kind of the IFIs is called an (*S, N*)-implication (or an *S*-implication). These are implications with the truth value

$$I(V(x), V(y)) = S(N(V(x)), V(y)),$$

where *S* is some *s*-norm and *N* is some IF negation operator. In this case the *s*-norm *S* must be an intuitionistic counterpart of the classical *s*-norm (see e.g. [17]).

2   Main results

We introduce now a new weak intuitionistic fuzzy implication $\rightarrow$. The given below implication is a result of using of the operation $\triangle$ playing the role of the *s*-norm in the (*S, N*)-implication. Owing to this fact, we will call this implication the *implication based on $\triangle$ operation*. The negation is in this case the classical negation $\neg$. Symbolically, we write: $V(x \rightarrow y) = V(\neg x) \triangle V(y)$. We can formulate the following Theorem 1.

Theorem 1. *Let $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$ be the truth-values of the variables x and y, respectively, and a, b, c, d, a + b, c + d $\in$ [0, 1]. The intuitionistic logical connective $\rightarrow$ with the truth-value*

$$V(x \rightarrow y) = \left\langle \frac{b+c}{a+b+c+d}, \frac{a+d}{a+b+c+d} \right\rangle$$

is a weak intuitionistic fuzzy implication (WIFI).

Proof. We start with a preliminary note.

The pair $\left\langle \dfrac{b+c}{a+b+c+d}, \dfrac{a+d}{a+b+c+d} \right\rangle$ is an IFP because:

$1^0)$ $0 \le \dfrac{b+c}{a+b+c+d} \le 1,$

$2^0)$ $0 \le \dfrac{a+d}{a+b+c+d} \le 1,$

$3^0)$ $0 \le \dfrac{b+c}{a+b+c+d} + \dfrac{a+d}{a+b+c+d} = 1 \le 1.$

The connective $\to$ fulfills the conditions (WIFI 1)–(WIFI 5) because of the following reasoning:

(WIFI 1) Let $V(y) = \langle c, d \rangle$. If $\langle a_1, b_1 \rangle = V(x_1) \preceq V(x_2) = \langle a_2, b_2 \rangle$, then $a_1 \le a_2$ and $b_1 \ge b_2$ and $a_1 b_2 \le a_2 b_1$.

Therefore:

$$c(a_2 - a_1) + d(b_1 - b_2) + (a_2 b_1 - a_1 b_2) \ge 0,$$

so

$$a_2 b_1 + db_1 + ca_2 - a_1 b_2 - db_2 - ca_1 \ge 0,$$
$$a_2 b_1 + db_1 + ca_2 \ge a_1 b_2 + db_2 + ca_1,$$
$$a_2 b_1 + db_1 + ca_2 + b_1 b_2 + cb_1 + cb_2 + c^2 + cd \ge$$
$$\ge a_1 b_2 + db_2 + ca_1 + b_1 b_2 + cb_1 + cb_2 + c^2 + cd,$$
$$(b_1 + c)(a_2 + b_2 + c + d) \ge (b_2 + c)(a_1 + b_1 + c + d),$$

and finally

$$\frac{b_1 + c}{a_1 + b_1 + c + d} \ge \frac{b_2 + c}{a_2 + b_2 + c + d}.$$

In the same manner, we can check the inequality

$$\frac{a_1 + d}{a_1 + b_1 + c + d} \le \frac{a_2 + d}{a_2 + b_2 + c + d}.$$

Therefore,

$$\left\langle \frac{b_1 + c}{a_1 + b_1 + c + d}, \frac{a_1 + d}{a_1 + b_1 + c + d} \right\rangle \succeq \left\langle \frac{b_2 + c}{a_2 + b_2 + c + d}, \frac{a_2 + d}{a_2 + b_2 + c + d} \right\rangle,$$

hence $V(x_1 \to y) \succeq V(x_2 \to y)$, and the proof of (WIFI 1) is completed.

(WIFI 2) Let $V(x) = \langle a, b \rangle$. If $\langle c_1, d_1 \rangle = V(y_1) \preceq V(y_2) = \langle c_2, d_2 \rangle$, therefore $c_1 \le c_2$ and $d_1 \ge d_2$ and $c_1 d_2 \le c_2 d_1$.

Therefore

$$b(d_2 - d_1) + a(c_1 - c_2) + (c_1 d_2 - d_1 c_2) \le 0,$$

so

$$bd_2 + ac_1 + c_1 d_2 - bd_1 - ac_2 - d_1 c_2 \le 0,$$
$$bd_2 + ac_1 + c_1 d_2 \le bd_1 + ac_2 + d_1 c_2,$$
$$bd_2 + ac_1 + c_1 d_2 + ab + b^2 + bc_1 + bc_2 + c_1 c_2 \le$$
$$\le bd_1 + ac_2 + d_1 c_2 + ab + b^2 + bc_1 + bc_2 + c_1 c_2,$$
$$(b + c_1)(a + b + c_2 + d_2) \le (b + c_2)(a + b + c_1 + d_1),$$

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and finally

$$\frac{b+c_1}{a+b+c_1+d_1} \le \frac{b+c_2}{a+b+c_2+d_2} \ .$$

In the same manner, we can check the inequality

$$\frac{a+d_1}{a+b+c_1+d_1} \ge \frac{a+d_2}{a+b+c_2+d_2} \ .$$

Therefore,

$$\left\langle \frac{b+c_1}{a+b+c_1+d_1}, \frac{a+d_1}{a+b+c_1+d_1} \right\rangle \preceq \left\langle \frac{b+c_2}{a+b+c_2+d_2}, \frac{a+d_2}{a+b+c_2+d_2} \right\rangle,$$

hence $V(x \to y_1) \preceq V(x \to y_2)$, and the proof of (WIFI 2) is completed.

(WIFI 3) Let $V(y) = \langle c, d \rangle$. It is, $V(\underline{0} \to y) = \left\langle \frac{c+1}{c+d+1}, \frac{d}{c+d+1} \right\rangle$. Since $\frac{c+1}{c+d+1} \ge \frac{d}{c+d+1}$

is equivalent for the inequality $c + 1 \ge d$, which holds for $c, d \in [0, 1]$, therefore, $\underline{0} \to y$ is an IFT.

(WIFI 4) Let $V(x) = \langle a, b \rangle$. It is $V(x \to \underline{1}) = \left\langle \frac{b+1}{a+b+1}, \frac{a}{a+b+1} \right\rangle$. Since $\frac{b+1}{a+b+1} \ge \frac{a}{a+b+1}$ is

equivalent for the inequality $b + 1 \ge a$, and this holds for $a, b \in [0, 1]$, therefore $x \to \underline{1}$ is an IFT.

(WIFI 5) It is $V(\underline{1} \to \underline{0}) = \langle 0, 1 \rangle = V(\underline{0})$, therefore $\underline{1} \to \underline{0}$ is an IFcT.

This completes the proof. □

The implication $\to$ fulfills the condition (IFI 4) and (IFI 6) of the Definition 2 but it does not fulfill the (IFI 3) and (IFI 5). It holds only (IFI 3′) and (IFI 5′) in the form:

(IFI 3′) $V(\underline{1} \to \underline{1}) = \langle 0.5, 0.5 \rangle$,

(IFI 5′) $V(\underline{0} \to \underline{0}) = \langle 0.5, 0.5 \rangle$.

Moreover, it holds that $V(x \to x) = \langle 0.5, 0.5 \rangle$ for any variable x.

In the literature[1] on fuzzy implications (not necessarily intuitionistic fuzzy implications), in addition to (WIFI 1)–(WIFI 5) or (IFI 1)–(IFI 6), the following axioms are further postulated:

(IFI 7) $V(\underline{1} \Rightarrow y) = V(y)$,

(IFI 8) $V(x \Rightarrow x) = V(\underline{1})$,

(IFI 9) $V(x \Rightarrow (y \Rightarrow z)) = V(y \Rightarrow (x \Rightarrow z))$

(IFI 10) $V(x \Rightarrow y) = V(\underline{1})$ iff $V(x) \preceq V(y)$,

(IFI 11) $V(x \Rightarrow y) = V(N(y) \Rightarrow N(x))$, while N is a negation,

where x, y, z are some variables with truth-values $V(x) = \langle a, b \rangle$, $V(y) = \langle c, d \rangle$, $V(z) = \langle e, f \rangle$ and $a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1]$, and $\Rightarrow$ is an implication.

Theorem 2. *The implication $\to$*

   a)   *does not satisfy (IFI 7), (IFI 8), and (IFI 9),*

   b)   *does not satisfy (IFI 10), but if $V(x \to y) = V(\underline{1})$, then $V(x) \preceq V(y)$,*

   c)   *satisfies (IFI 11) with $N = \neg$.*

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[1] Various authors give these and other axioms following [23, pp. 308, 310].

Proof.

a) We check consequently for (IFI 7), (IFI 8) and (IFI 9):

(IFI 7) $V(\underline{1} \to y) = \left\langle \dfrac{c}{1+c+d}, \dfrac{1+d}{1+c+d} \right\rangle \neq \langle c, d \rangle.$

(IFI 8) $V(x \to x) = \langle 0.5, 0.5 \rangle \neq \langle 1, 0 \rangle.$

(IFI 9)   It is easy to show that the equality does not hold (in generally) by a counterexample: $a = d = f = 1$, $b = c = e = 0$.

b) We perform the check:

(IFI 10)   If $V(x \to y) = V(\underline{1})$, i.e., $\dfrac{b+c}{a+b+c+d} = 1$ and $\dfrac{a+d}{a+b+c+d} = 0$, so that $a + d = 0$,

which holds only for $a = d = 0$, therefore $V(x) = \langle 0, b \rangle \overset{\leq}{} \langle c, 0 \rangle = V(y)$. In the other direction, if $V(x) \preceq V(y)$, i.e., $a \leq c$ and $b \geq d$, then not necessarily it must hold that $V(x \to y) = V(\underline{1})$. Counter-example: $a = b = c = d = 0.5$.

c) We perform the check:

(IFI 11)   If $V(N(x)) = \langle b, a \rangle$, $V(N(y)) = \langle d, c \rangle$, then

$$V(N(y) \to N(x)) = \left\langle \frac{b+c}{a+b+c+d}, \frac{a+d}{a+b+c+d} \right\rangle = V(x \to y).$$

This completes the proof.   □

Remarks:

R1. The implication $\to$ does not satisfy (IFI 7), however, if $\underline{1} \to y$ is an IFT, then y is an IFT, and if y is an IFcT, then $\underline{1} \to y$ is an IFcT.

R2. The implication $\to$ does not satisfy (IFI 8), however, $x \to x$ is an IFT.

R3. The implication $\to$ does not satisfy (IFI 10), however, if $V(x) \preceq V(y)$, then $x \to y$ is an IFT.

It is also easy to check that the implication $\to$ does not satisfy all of the classical (two-valued) logic axioms. Namely, it is $V(\underline{0} \to \underline{0}) = V(\underline{1} \to \underline{1}) = \langle 0.5, 0.5 \rangle \neq V(\underline{1})$, although $V(\underline{1} \to \underline{0}) = \langle 0, 1 \rangle = V(\underline{0})$ and $V(\underline{0} \to \underline{1}) = \langle 1, 0 \rangle = V(\underline{1})$. However, we should notice that $\underline{0} \to \underline{0}$ and $\underline{1} \to \underline{1}$ are IFTs.

As we can see, therefore, the implication $\to$ is not a generalization of the classical implication.

There exist two basic rules of inference: *Modus Ponens* and *Modus Tollens*. These are the tautologies, given in the two-valued logic in the form

$$(p \wedge (p \Rightarrow q)) \Rightarrow q$$

and

$$((p \Rightarrow q) \wedge N(q)) \Rightarrow N(p),$$

respectively.

We assume that the *Modus Ponens* in the IFL-case is as follows:

if x is an IFT and $x \Rightarrow y$ is an IFT, then y is an IFT.

Similarly, we assume the *Modus Tollens* rule in the IFL-case as follows:

if $x \Rightarrow y$ is an IFT and y is an IFcT then x is an IFcT.

Theorem 3. *The implication $\rightarrow$*
   a)   satisfies Modus Ponens in the IFL-case,
   b)   satisfies Modus Tollens in the IFL-case.

Proof. Let $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$.

a)   Let x and $x \rightarrow y$ be IFTs. Therefore, $a \geq b$ and $\dfrac{b+c}{a+b+c+d} \geq \dfrac{a+d}{a+b+c+d}$. Further, $a - b \geq$

   0 and $b + c \geq a + d$. So, $0 \leq a - b$ and $a - b \leq c - d$. It follows that $0 \leq c - d$. Hence $c \geq d$
   and y is an IFT.

b)   Let $x \rightarrow y$ be an IFT and y be an IFcT. Then, $\dfrac{b+c}{a+b+c+d} \geq \dfrac{a+d}{a+b+c+d}$, and $c \leq d$.

   Therefore, $b + c \geq a + d$ and $d - c \geq 0$. So, $0 \leq d - c$ and $d - c \leq b - a$. It follows that $0 \leq b - a$.
   Hence, $a \leq b$ and x is an IFcT.

This completes the proof.   □

Remarks:

R4. For $V(x) = \langle 1, 0 \rangle$, if $x \rightarrow y$ would be an IFT, i.e., if it would hold that

$$\frac{b+c}{a+b+c+d} \geq \frac{a+d}{a+b+c+d},$$

then we would have $c \geq 1 + d$, then $V(y) = \langle 1, 0 \rangle$.

R5. If $x \rightarrow y$ would be an IFT, i.e., if

$$\frac{b+c}{a+b+c+d} \geq \frac{a+d}{a+b+c+d},$$

then for $V(y) = \langle 0, 1 \rangle$ holds the inequality $b \geq a + 1$, therefore $V(x) = \langle 0, 1 \rangle$.

   One of the fundamental tautologies of classical logic is the relationship between the implication and negation. This relationship says that the truth-value of negation of the variable x is equal to the value of the logical implications of the antecedent x and the consequent False. Symbolically, this tautology is written in the classical logic in the form of $N(x) \Leftrightarrow (x \Rightarrow 0)$. Using this relationship, we can, for every intuitionistic fuzzy implication, designate a corresponding negation, called a *generated (induced) negation*.

Theorem 4. *Let $V(x) = \langle a, b \rangle$. The negation $N_\triangle$ generated by the implication $\rightarrow$ is expressed by formula:*

$$V(N_\triangle(x)) = \left\langle \frac{b}{a+b+1}, \frac{a+1}{a+b+1} \right\rangle.$$

Proof. It follows from the definition of the $\rightarrow$ implication.   □

Remarks:

R6. $N_\triangle(\underline{0}) = \langle 0.5, 0.5 \rangle$,
   $N_\triangle(\underline{1}) = \langle 0, 1 \rangle = V(\underline{0})$,
   $N_\triangle(\underline{FI}) = \langle 0, 1 \rangle = V(\underline{0})$.

R7. For any variable x the negation $N_\triangle(x)$ is an IFcT.

R8. The negation $N_\triangle(x)$ is not involutive because

$$V(N_\triangle(N_\triangle(x))) = \left\langle \frac{a+1}{2(a+b+1)}, \frac{a+2b+1}{2(a+b+1)} \right\rangle \neq \langle a, b \rangle = V(x).$$

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The first equality in the remark R6 shows that the above negation does not fulfill the basic property of negations in form $V(N(\underline{0})) = V(\underline{1})$, however $N(\underline{0})$ is an IFT. The property presented in remark R7 should not be satisfied because the negation of an IFT should be an IFcT and the negation of an IFcT should be an IFT. For this reason, the negation $N_\triangle$ should be carefully used in different applications.

3   Conclusion

In the paper the new fuzzy intuitionistic implication based on the operation $\triangle$ is presented together with its basic properties. The implication may be the subject of further research, both in terms of its properties or with regards to comparisons with other intuitionistic fuzzy implications, and possible applications. Possible applications, for example, can be related to fuzzy control, reasoning with incomplete or uncertain information, or multiple criteria decision making, especially with varying degrees of criteria importance.

References

[1]   Atanassov, K.T. (1983/2016). Intuitionistic fuzzy sets. VII ITKR's Sci. Session, Sofia, (June 1983) (Deposed in Central Sci. – Techn. Library of Bulg. Acad. of Sci., Hπ 1697/84) (in Bulg.). Reprint and English version in: International Journal Bioautomation, Vol. 20, supl. 1, 1–6.

[2]   Atanassov, K.T. (2017). *Intuitionistic Fuzzy Logics.* Springer. Cham.

[3]   Atanassov, K.T. (2021). Third Zadeh's Intuitionistic Fuzzy Implication. *Mathematics*, Vol. 9, No. 6, 619.

[4]   Atanassov, K.T., Angelova, N., & Atanassova, V. (2021). On an Intuitionistic Fuzzy Form of the Goguen's Implication. *Mathematics*, Vol. 9, No. 6, 676.

[5]   Atanassov, K.T., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, Vol. 19, No. 3, 1–13.

[6]   Atanassova, L. (2009). A new intuitionistic fuzzy implication. *Cybernetics and Information Technologies*, Vol. 9, No 2, 21–25.

[7]   Atanassova, L. (2012). On two modifications of the intuitionistic fuzzy implication $\rightarrow_{@}$. *Notes on Intuitionistic Fuzzy Sets*, Vol. 18, No. 2, 26–30.

[8]   Atanassova, L. (2013). On the modal form of the intuitionistic fuzzy implications $\rightarrow'_{@}$ and $\rightarrow''_{@}$. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, Vol. 10, 5–11.

[9]   Atanassova, L. (2015). Remark on Dworniczak's intuitionistic fuzzy implications. Part 1. *Notes on Intuitionistic Fuzzy Sets*, Vol. 21, No. 3, 18–23.

[10] Atanassova, L. (2016). Remark on Dworniczak's intuitionistic fuzzy implications. Part 2. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, Vol. 12, 61–67.

[11] Atanassova, L. (2016). Remark on Dworniczak's intuitionistic fuzzy implications. Part 3. *Notes on Intuitionistic Fuzzy Sets*, Vol. 22, No. 31, 1–6.

[12] Atanassova, L. (2020). A new operator over intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, Vol. 24, No. 1, 23–28.

[13] Atanassova, L., & Dworniczak, P. (2021). On the operation $\triangle$ over intuitionistic fuzzy sets. *Mathematics*, in print.

[14] Baczyński, M., & Jayaram, B. (2008). *Fuzzy implications*. Springer, Berlin.

[15] Cornelis, C., & Deschrijver, G. (2001). The compositional rule of inference in an intuitionistic fuzzy logic setting (cited also as: The compositional rule of inference in an intuitionistic fuzzy logic framework). In: Striegnitz, K., (ed.), *Proceedings of ESSLLI 2001*, Student Session, Kluwer Academic Publishers, 83–94.

[16] Cornelis, C., Deschrijver, G., Cock, M., & Kerre, E.E. (2002). Intuitionistic fuzzy relational calculus. *Proceedings of the First International IEEE Symposium "Intelligent Systems"*, September 2002, Varna, Vol. 1, 340–345.

[17] Cornelis, C., Deschrijver, G., & Kerre, E.E. (2002). Classification of intuitionistic fuzzy implicators: an algebraic approach. *Proceedings of the FT&T'02, 8th International Conference on Fuzzy Theory and Technology*, March 9–12, 2002, Durham, North Carolina, USA, 105–108.

[18] Cornelis, C., Deschrijver, G., & Kerre, E.E. (2004). Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application. *International Journal of Approximate Reasoning*, Vol. 35, No.1, 55–95.

[19] Dworniczak, P. (2011). Inclusion of the intuitionistic fuzzy sets based on some weak intuitionistic fuzzy implications. *Cybernetics and Information Technologies*, Vol. 11, No. 3, 12–22.

[20] Dworniczak, P. (2010). Some remarks about the L. Atanassova's paper "A new intuitionistic fuzzy implication". *Cybernetics and Information Technologies*, Vol.10, No 3, 3–9.

[21] Dworniczak, P. (2010). On one class of intuitionistic fuzzy implications. *Cybernetics and Information Technologies*, Vol. 10, No. 4, 13–21.

[22] Dworniczak, P. (2011). On some two-parametric intuitionistic fuzzy implications. *Notes on Intuitionistic Fuzzy Sets*, Vol. 17, No. 2, 8–16.

[23] Klir, G., & Yuan, B., (1995). *Fuzzy sets and fuzzy logic*. Prentice Hall, New Jersey.

[24] Liu, H.-W., & Wang, G.-J., (2006). A note on implicators based on binary aggregation operators in interval-valued fuzzy set theory. *Fuzzy Sets and Systems*, Vol. 157, No. 24, 3231–3236.

[25] Wang, Z., Xu, Z., Liu, S., & Yao, Z. (2014). Direct clustering analysis based on intuitionistic fuzzy implication. *Applied Soft Computing*, Vol. 23, 1–8.

[26] Zhou, L., Wu, W.-Z., Zhang, & W.-X., (2009). On characterization of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators. *Information Sciences*, Vol. 179, No. 7, 883–898.