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λ-Statistical convergence of order α in intuitionistic fuzzy n-normed spaces

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Abstract: In the present paper, we introduce the notion $[V, \lambda](I)$-summability and I_λ-statistical convergence of order α in the framework of intuitionistic fuzzy n-normed space, briefly IFnNS, also we examine the relationship between these classes.
Keywords: I-statistical convergence, I_λ-statistical convergence of order α, I–$[V, \lambda]$-summability, intuitionistic fuzzy n-normed space.
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1 Introduction and Preliminaries

Following the introduction of fuzzy set theory by Zadeh [1] there has been extensive research to find applications and fuzzy analogues of the classical theories. The theory of intuitionistic fuzzy sets was introduced by Atanassov [2] it has been extensively used in decision-making problems [3]. The concept of an intuitionistic fuzzy metric space was introduced by Park [4]. Further, Saadati and Park [5] gave the notion of an intuitionistic fuzzy normed space. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in ([6, 7, 8]). Using the concepts of n-normed linear spaces and fuzzy normed linear spaces; in [9] fuzzy n-normed linear space and in [10] intuitionistic fuzzy n-normed linear space have been defined. Also in [11] the notions of lacunary statistical convergence and lacunary statistical Cauchy sequence have been introduced.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [12], and Schoenberg [13], independently for the real sequences. Later on it was further investigated from sequence point of view and linked with the summability theory by Fridy [14], Salat [15] and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows: The natural density of a subset K of $\mathbb{N}$ is denoted by $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left| \{k \in K : k \le n\} \right|, \tag{1}$$

where the vertical bar denotes the cardinality of the respective set.

Recently, Mursaleen [16] studied the concept of statistical convergence of sequences in random 2-normed space. Quite recently, Savaş [17] introduced λ-statistical convergence theorem in random 2-normed space and in [18] Savaş proved some theorems in intuitionistic fuzzy 2-normed space using λ-statistical convergence.

The concepts of *I*-statistical convergence, *I*-lacunary statistical convergence, and I_λ-statistical convergence have been introduced by using ideals and have been investigated their properties in [19, 20]. Savaş and Gürdal [21] also studied the concept of I_λ-statistical convergence with respect to the intuitionistic fuzzy normed space (μ, ν). Also Savaş [25] introduced $[V, \lambda](I)$-summability and I_λ-statistical convergence of order $\alpha, 0 < \alpha \le 1,$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)_n$ and investigated some properties of these classes.

In this paper, we introduce $[V,\lambda](I)$-summability and I_λ-statistical convergence of order $\alpha, 0 < \alpha \le 1,$ in the setting of intuitionistic fuzzy normed space.

Firstly, we recall some notations and definitions which we need them in the sequel.

Definition 1.1 [22] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associate and commutative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$,

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2. [22] A binary operation $\lozenge : [0,1] \times [0,1] \to [0,1]]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\lozenge$ is associate and commutative,

(ii) $\lozenge$ is continuous,

(iii) $a \lozenge 0 = a$ for all $a \in [0,1]$,

(iv) $a \lozenge b \le c \lozenge d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

For example, we can give $a * b = ab$, $a * b = \min\{a, b\}$, $a \lozenge b = \min\{a + b, 1\}$ and $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0,1]$.

Definition 1.3. [10] An intuitionistic fuzzy n-normed space (or) in short IFnNS is an object of the form $A=\{X,\mu(x,t),\nu(x,t):x=(x_1,x_2,...,x_n)\in X^n\}$, where X is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-co-norm and μ, ν are fuzzy sets on $X^n\times(0,\infty)$; μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1,x_2,...,x_n,t)\in X^n\times(0,\infty)$ $X\times(0,\infty)$ satisfying the following conditions:

i. $\mu(x,t)+\nu(x,t)\leq 1$,

ii. $\mu(x,t)>0$,

iii. $\mu(x_1,x_2,...,x_n,t)=1$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

iv. $\mu(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1,x_2,...,x_n$.

v. $\mu(x_1,x_2,...,cx_n,t)=\mu\left(x_1,x_2,...,x_n,\frac{t}{|c|}\right)$ for each $c\neq 0$,

vi. $\mu(x_1,x_2,...,x_n,t)*\mu(x_1,x_2,...,x_n',s)\leq\mu(x_1,x_2,...,x_n+x_n',t+s)$,

vii. $\mu(x,.):(0,\infty)\to[0,1]$ is continuous,

viii. $\nu(x,t)>0$,

ix. $\nu(x_1,x_2,...,x_n,t)=0$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

x. $\nu(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1,x_2,...,x_n$.

xi. $\nu(x_1,x_2,...,cx_n,t)=\nu\left(x_1,x_2,...,x_n,\frac{t}{|c|}\right)$ for each $c\neq 0$,

xii. $\nu(x_1,x_2,...,x_n,t)\Diamond\nu(x_1,x_2,...,x_n',s)\geq\nu(x_1,x_2,...,x_n+x_n',t+s)$

xiii. $\nu(x,.):(0,\infty)\to[0,1]$ is continuous.

For convenience we denote the intuitionistic fuzzy *n*-normed space (or) in short IFnNS by $(X,\mu,\nu,*,\Diamond)$ and also intuitionistic fuzzy n-norm by $(\mu,\nu)_n$. As a standard example, we give the following.

Example 1.4. [11] Let $(X,\|\bullet,...,\bullet\|)$ be an *n*-normed space, where $X=\mathbb{R}$. Define $a*b=\min\{a,b\}$ and $a\Diamond b=\max\{a,b\}$ for all $a,b\in[0,1]$, for all $x=(x_1,x_2,...,x_n)\in X$ and every $t>0$;

$$\mu(x_1,x_2,...,x_n,t)=e^{\frac{-\|x_1,x_2,...,x_n\|}{t}},\quad \nu(x_1,x_2,...,x_n,t)=1-e^{\frac{-\|x_1,x_2,...,x_n\|}{t}}.$$

Then $(X,\mu,\nu,*,\Diamond)$ is an intuitionistic fuzzy n-normed space.

Definition 1.5. [10] Let $(X,\mu,\nu,*,\Diamond)$ be an IFnNS. A sequence $x=(x_{n_k})\in(X,\mu,\nu,*,\Diamond)$ is said to be convergent to $L\in X$ with respect to the intuitionistic fuzzy n-norm $(\mu,\nu)_n$ if, for every $\varepsilon>0$ and $t>0$, there exists a positive integer n_0 such that $\mu(x_1,x_2,...,x_{n-1},x_{n_k}-L,t)>1-\varepsilon$

and $\nu\left(x_1, x_2, ..., x_{n-1}, x_{n_k} - L,\ t\right) < \varepsilon$ for all $k \geq n_0$. It is denoted by $(\mu, \nu)_n - \lim x_{n_k} = L$ or $x_{n_k} \xrightarrow{(\mu,\nu)_n} L$ as $k \to \infty$.

Definition 1.6. [10] Let $(\mathrm{X},\ \mu,\ \mathrm{v}, *, \Diamond)$ be an IFnNS. A sequence $x = \left(x_{n_k}\right) \in \left(X, \mu, \nu, *, \Diamond\right)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy *n*-norm $(\mu,\ \nu)_n$ if, for each $\varepsilon > 0$ and $t > 0$, there exists a positive integer m_0 such that $\mu\ \left(x_1, x_2, ..., x_{n-1}, x_{n_p} - x_{n_q},\ t\right) > 1 - \varepsilon$ and $\nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - x_{n_q},\ t\right) < \varepsilon$ whenever $p, q \geq m_0$.

Before proceeding further, we should recall some notation on the ideal.
A family $I \subset 2^Y$ of subsets of a nonempty set *Y* is said to be an ideal in *Y* if

i. $\varnothing \in I$;
ii. $A, B \in I$ imply $A \cup B \in I$;
iii. $A \in I,\ B \subset A$ imply $B \in I$.

A proper ideal *I* is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

i. $\varnothing \notin F$;
ii. $A, B \in F$ imply $A\ \cap\ B \in F$;
iii. $A \in F,\ A \subset B$ imply $B \in F$.

If I is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(I) = \{M \subset N : \exists A \in I : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

2 *I*$_\lambda$-statistical convergence in IFnNS

Definition 2.1. Let $(\mathrm{X},\ \mu,\ \mathrm{v}, *, \Diamond)$ be an IFnNS. A sequence $x = \left(x_{n_k}\right)$ is said to be I-statistically convergent of order α to $L \in X$ with respect to $(\mu,\ \nu)_n$, where $0 < \alpha \leq 1$, if for every $\varepsilon > 0$, $\delta > 0$ and $t > 0$;

$$\left\{k \in \mathbb{N} : \frac{1}{k^\alpha}\left|\left\{p \leq k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

which denotes as $x_{n_k} \to L\left(S^\alpha\left(\mathrm{I}\right)^{(\mu,\nu)_n}\right)$ or $S^\alpha\left(\mathrm{I}\right)^{(\mu,\nu)_n} - \lim x_{n_k} = L$.

Remark 2.2. For $I = I_{fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}, S^\alpha\left(\mathrm{I}\right)^{(\mu,\nu)_n} -$convergence coincides with statistically convergence of order α with respect to $(\mu,\ \nu)_n$. For an arbitrary ideal *I* and for $\alpha = 1$ it coincides with *I*-statistical convergence with respect to $(\mu,\ \nu)_n$, [23]. When $I = I_{fin}$ and α=1 it becomes only statistically convergence with respect to $(\mu,\ \nu)_n$, [21].

Let $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The collection of such sequences λ will be denoted by Δ.

We define the generalized de la Vallee-Pousin mean of order α by

$$t_n(x) = \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 2.3. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. A sequence $x = (x_{n_k})$ is said to be $[V, \lambda](I)$-statistically convergent of order α to $L \in X$ with respect to $(\mu, \nu)_n$, where $0 < \alpha \leq 1$, if for every $\varepsilon > 0$ and $t > 0$;

$$\left\{k \in \mathbb{N} : \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \geq \varepsilon\right\} \in I,$$

which is denoted as $[V, \lambda]^\alpha (I)^{(\mu,\nu)_n} - \lim x = L.$

Definition 2.4. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. A sequence $x = (x_{n_k})$ is said to be I_λ-statistically convergent of order α or $S_\lambda^\alpha(\mathrm{I})^{(\mu,\nu)_n}$-convergent to $L \in X$ with respect to $(\mu, \nu)_n$ where $0 < \alpha \leq 1$, if for every $\varepsilon > 0$ and $t > 0$;

$$\left\{k \in \mathbb{N} : \frac{1}{\lambda_k^\alpha}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

which is denoted as $S_\lambda^\alpha(\mathrm{I})^{(\mu,\nu)_n} - \lim x = L$ or $x_{n_k} \to L\left(S_\lambda^\alpha(\mathrm{I})^{(\mu,\nu)_n}\right)$.

Remark 2.5. For $I = I_{fin}, S_\lambda^\alpha(\mathrm{I})^{(\mu,\nu)_n}$ – convergence with respect to $(\mu, v)_n$ coincides with λ– statistically convergence of order α with respect to $(\mu, v)_n$. For an arbitrary ideal I and for $\alpha = 1$ it coincides with I_λ-statistical convergence with respect to $(\mu, v)_n$, [21]. Finally $I = I_{fin}$ and α=1 it becomes λ-statistically convergence with respect to $(\mu, v)_n$, [24].

Theorem 2.6. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. Let $\lambda = (\lambda_k) \in \Delta$. Then $x_{n_k} \to L\left([V, \lambda]^\alpha (I)^{(\mu,\nu)_n}\right) \Rightarrow x_{n_k} \to L\left(S_\lambda^\alpha(\mathrm{I})^{(\mu,\nu)_n}\right)$.

Proof. For every $\varepsilon > 0$ and $t > 0$, let $x_{n_k} \to L\left([V, \lambda]^\alpha (I)^{(\mu,\nu)_n}\right)$. We have

$$\sum_{p \in I_k} \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right)$$

$$\geq \sum_{\substack{p \in I_k \,\&\, \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) < 1 - \varepsilon \\ \text{or } \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) > \varepsilon}} \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right)$$

$$\geq \varepsilon\left|\left\{p \in \mathbb{N} : \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \geq \varepsilon\right\}\right|.$$

Then for a given $\delta > 0$,

$$\frac{1}{\lambda_k^{\alpha}}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right| \ge \delta$$

$$\Rightarrow \frac{1}{\lambda_k^{\alpha}} \sum_{p \in I_k} \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \le (1-\varepsilon)\delta \quad \text{or} \quad \frac{1}{\lambda_k^{\alpha}} \sum_{p \in I_k} \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \ge \varepsilon\delta,$$

which implies that

$$\left\{k \in \mathbb{N} : \frac{1}{\lambda_k^{\alpha}}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right| \ge \delta\right\}$$

$$\subset \{k \in \mathbb{N} :$$

$$\frac{1}{\lambda_k^{\alpha}}\left\{\sum_{p \in I_k} \mu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \le (1-\varepsilon) \quad \text{or} \quad \sum_{p \in I_k} \nu\left(x_1, x_2, ..., x_{n-1}, t_{n_k}(x) - L; t\right) \ge \varepsilon\right\} \ge \varepsilon\delta\right\}.$$

Since $x_{n_k} \to L\left([V, \lambda]^{\alpha}(I)^{(\mu,\nu)_n}\right)$, we see that $x_{n_k} \to L\left(S_{\lambda}^{\alpha}(\mathrm{I})^{(\mu,\nu)_n}\right)$, so this completes the proof. □

To show that $\left(S_{\lambda}^{\alpha}(\mathrm{I})^{(\mu,\nu)_n}\right) \subsetneq \left([V, \lambda]^{\alpha}(I)^{(\mu,\nu)_n}\right)$, take a fixed $A \in I$. Define $x = (x_k)$ by

$$x_k = \begin{cases} ku, & \text{for } n - \left[\sqrt{\lambda_n^{\alpha}}\right] + 1 \le k \le n, n \notin A \\ ku, & \text{for } n - \lambda_n + 1 \le k \le n, n \in A \\ \theta, & \text{otherwise.} \end{cases}$$

where $u \in X$ is a fixed element with $\|u\| = 1$, and θ is the null element of X. Then $x \notin m(X)$ and for every $\varepsilon > 0\ (0 < \varepsilon < 1)$ since

$$\frac{1}{\lambda_n^{\alpha}}\left|\left\{k \in \mathbb{N} : \left\|x_1, x_2, ..., x_{n-1}, x_{n_k} - \theta\right\| \ge \varepsilon\right\}\right| = \frac{\left[\sqrt{\lambda_n^{\alpha}}\right]}{\lambda_n^{\alpha}} \to 0$$

as $n \to \infty$ and $n \notin A$, so for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}}\left|\left\{k \in \mathbb{N} : \left\|x_1, x_2, ..., x_{n-1}, x_{n_k} - \theta\right\| \ge \varepsilon\right\}\right| \ge \delta\right\} \subset A \cup \{1, 2, ..., m\}$$

for some $m \in \mathbb{N}$. Since I is admissible so it follows that $x_{n_k} \to \theta\left(S_{\lambda}^{\alpha}(\mathrm{I})^{(\mu,\nu)_n}\right)$. Obviously

$$\frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left\|x_1, x_2, ..., x_{n-1}, x_{n_k} - \theta\right\| \to \infty\ (n \to \infty)$$

i.e. $x_{n_k} \nrightarrow \theta\left([V, \lambda]^{\alpha}(I)^{(\mu,\nu)_n}\right)$. Note that if $A \in I$ is infinite then $x_{n_k} \nrightarrow \theta\left(S_{\lambda}^{(\mu,\nu)_n}\right)^{\alpha}$. This example also shows that $I - S_{\lambda}^{(\mu,\nu)_n}$-statistical convergence of α is more general than $S_{\lambda}^{(\mu,\nu)_n}$-statistical convergence of order α.

Theorem 2.7. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. If $\liminf_{k\to\infty} \frac{\lambda_k^\alpha}{k^\alpha} > 0$, then $S^\alpha(I)^{(\mu,\nu)_n} \subset S_\lambda^\alpha(I)^{(\mu,\nu)_n}$.

Proof. For fixed $\varepsilon > 0$ and $t > 0$, we have

$$\frac{1}{k^\alpha}\left|\left\{p \le k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right|$$

$$\ge \frac{1}{k^\alpha}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right|$$

$$= \frac{\lambda_k^\alpha}{k^\alpha} . \frac{1}{\lambda_k^\alpha}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right|.$$

If $\lim_{k\to\infty} \frac{\lambda_k^\alpha}{k^\alpha} = \alpha$, then from the definition the set $\left\{k \in \mathbb{N} : \frac{\lambda_k^\alpha}{k^\alpha} < \frac{\alpha}{2}\right\}$ is finite. For every $\delta > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\lambda_k^\alpha}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right| \ge \delta\right\}$$

$$\subset \left\{k \in \mathbb{N} : \frac{1}{k^\alpha}\left|\left\{p \in I_k : \mu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \le 1-\varepsilon \text{ or } \nu\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - L; t\right) \ge \varepsilon\right\}\right| \ge \frac{\alpha}{2}\delta\right\}$$

$$\bigcup \left\{k \in \mathbb{N} : \frac{\lambda_k^\alpha}{k^\alpha} < \frac{\alpha}{2}\right\}.$$

This completes the proof, as *I* is admissible the set on the right-hand side belongs to *I*. □

Theorem 2.8. Let $(X, \mu, v, *, \Diamond)$ be an IFnNS. If $\lambda \in \Delta$ be such that for a particular $\alpha, 0 < \alpha < 1, \lim_k \frac{k - \lambda_k}{k^\alpha} = 0$, then $S_\lambda^\alpha(I)^{(\mu,\nu)_n} \subset S^\alpha(I)^{(\mu,\nu)_n}$.

Proof. Let $\delta > 0$ be given. Since $\lim_k \frac{k - \lambda_k}{k^\alpha} = 0$, we can choose $m \in \mathbb{N}$ such that $\frac{k - \lambda_k}{k^\alpha} < \frac{\delta}{2}$, for all $k \ge m$. Now observe that, for $\varepsilon > 0$

$$\frac{1}{k^{\alpha}}\left|\left\{p\le k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|$$

$$=\frac{1}{k^{\alpha}}\left|\left\{p\le k-\lambda_k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|$$

$$+\frac{1}{k^{\alpha}}\left|\left\{p\in I_k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|$$

$$\le\frac{k-\lambda_k}{k^{\alpha}}+\frac{1}{k^{\alpha}}\left|\left\{p\in I_k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|$$

$$\le\frac{\delta}{2}+\frac{1}{\lambda_k^{\alpha}}\left|\left\{p\in I_k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|,$$

for all $k\ge m$. Hence

$$\left\{k\in\mathbb{N}:\frac{1}{k^{\alpha}}\left|\left\{p\le k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|\ge\delta\right\}$$

$$\subset\left\{k\in\mathbb{N}:\frac{1}{\lambda_k^{\alpha}}\left|\left\{p\in I_k:\mu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\le 1-\varepsilon \text{ or } \nu\left(x_1,x_2,...,x_{n-1},x_{n_p}-L;t\right)\ge\varepsilon\right\}\right|\ge\frac{\delta}{2}\right\}$$

$$\bigcup\{1,2,...,m\}.$$

If $S_{\lambda}^{\alpha}(I)^{(\mu,\nu)_n}-\lim x=L,$ then the set on the right-hand side belongs I and so the set on the left-hand side also belongs to *I*. This shows that $x=\left(x_{n_k}\right)$ is I-statistically convergent of order α to L with respect to intuitionistic fuzzy n-normed space $(\mu,\nu)_n$. □

Theorem 2.9. Let $(\text{X},\mu,\text{v},*,\Diamond)$ be an IFnNS such that $\frac{\varepsilon_k}{4}\Diamond\frac{\varepsilon_k}{4}<\frac{\varepsilon_k}{2}$ and $\left(1-\frac{\varepsilon_k}{4}\right)*\left(1-\frac{\varepsilon_k}{4}\right)>1-\frac{\varepsilon_k}{2}$. If X is a Banach space then $S_{\lambda}^{\alpha}(I)^{(\mu,\nu)_n}\cap l_{\infty}^{(\mu,\nu)_n}$ is a closed subset of $l_{\infty}^{(\mu,\nu)_n}$, where $l_{\infty}^{(\mu,\nu)_n}$ stands for the space of all bounded sequences of intuitionistic fuzzy *n*-norm $(\mu,\nu)_n$.

Proof. We first assume that $\left(x^j\right)\subset S_{\lambda}^{\alpha}(I)^{(\mu,\nu)_n}\cap l_{\infty}^{(\mu,\nu)_n}$, $0<\alpha\le 1,$ is a convergent sequence and it converges to $x\in l_{\infty}^{(\mu,\nu)_n}$. We need to show that $x\in S_{\lambda}^{\alpha}(I)^{(\mu,\nu)_n}\cap l_{\infty}^{(\mu,\nu)_n}$. Suppose that $x^j\to L_j\left(S_{\lambda}^{\alpha}(I)^{(\mu,\nu)_n}\right)$ for all $j\in\mathbb{N}$. Take a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of strictly decreasing positive numbers converging to zero. We can find a $j\in\mathbb{N}$ such that $\sup_k \nu\left(x_1,x_2,...,x_{n-1},x-x_n^k;t\right)<\frac{\varepsilon_j}{4}$ for all $m\ge j$ m. Choose $0<\delta<\frac{1}{5}$. Now let

$$A_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)=$$

$$\left\{p\in\mathbb{N}:\frac{1}{\lambda_p^{\alpha}}\left|\left\{k\in I_p:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)\le 1-\frac{\varepsilon_j}{4}\text{ or }\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)\ge\frac{\varepsilon_j}{4}\right\}\right|<\delta\right\}$$

belongs to $F(I)$ and

$$B_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)=\left\{p\in\mathbb{N}:\frac{1}{\lambda_p^{\alpha}}\left|\left\{\begin{array}{c}k\in I_p:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)\le 1-\frac{\varepsilon_j}{4}\text{ or}\\ \nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)\ge\frac{\varepsilon_j}{4}\end{array}\right\}\right|<\delta\right\}$$

belongs to $F(I)$. Since $A_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)\cap B_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)\in F(I)$ and $\varnothing\notin F(I)$, we can choose $p\in A_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)\cap B_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)$. Then

$$\frac{1}{\lambda_p^{\alpha}}\left|\left\{\begin{array}{l}k\in I_p:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)\le 1-\frac{\varepsilon_j}{4}\text{ or }\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)\ge\frac{\varepsilon_j}{4}\vee\\ \mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)\le 1-\frac{\varepsilon_j}{4}\text{ or }\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)\ge\frac{\varepsilon_j}{4}\end{array}\right\}\right|\le 2\delta<1.$$

Since $\lambda_p\to\infty$ and $A_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)\cap B_{(\mu,\nu)_n}\left(\varepsilon_1,\varepsilon_2,...,\varepsilon_{n_j};t\right)\in F(I)$ is finite, we can actually choose the above p so that $\lambda_p>5$. Hence there must exists a $k\in I_p$ for which we have simultaneously, $\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)>1-\frac{\varepsilon_j}{4},\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-L_j;t\right)<\frac{\varepsilon_j}{4}$ and $\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)>1-\frac{\varepsilon_j}{4},\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};t\right)<\frac{\varepsilon_j}{4}$. For a given $\varepsilon_j>0$ choose $\frac{\varepsilon_j}{2}$ such that $\left(1-\frac{\varepsilon_j}{2}\right)*\left(1-\frac{\varepsilon_j}{2}\right)>1-\varepsilon_j$ and $\frac{\varepsilon_j}{2}\Diamond\frac{\varepsilon_j}{2}<\varepsilon_j$. Then it follows that

$$\nu\left(x_1,x_2,...,x_{n-1},L_j-x_{n_k}^{j};t\right)\Diamond\nu\left(x_1,x_2,...,x_{n-1},L_{j+1}-x_{n_k}^{j+1};t\right)\le\frac{\varepsilon_j}{4}\Diamond\frac{\varepsilon_j}{4}<\frac{\varepsilon_j}{2}$$

and

$$\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-x_{n_k}^{j+1};t\right)\le\sup_j\nu\left(x_1,x_2,...,x_{n-1},x-x_{n_k}^{j};\frac{t}{2}\right)\Diamond\sup_j\nu\left(x_1,x_2,...,x_{n-1},x-x_{n_k}^{j+1};\frac{t}{2}\right)$$
$$\le\frac{\varepsilon_j}{4}\Diamond\frac{\varepsilon_j}{4}<\frac{\varepsilon_j}{2}.$$

Hence we have

$$\nu\left(x_1,x_2,...,x_{n-1},L_j-L_{j+1};t\right)\le\left[\nu\left(x_1,x_2,...,x_{n-1},L_j-x_{n_k}^{j};\frac{t}{3}\right)\Diamond\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j+1}-L_{j+1};\frac{t}{3}\right)\right]$$

$$\Diamond\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^{j}-x_{n_k}^{j+1};\frac{t}{3}\right)$$

$$\le\frac{\varepsilon_j}{2}\Diamond\frac{\varepsilon_j}{2}<\varepsilon_j,$$

and similarly $\mu\left(x_1,x_2,...,x_{n-1},L_j-L_{j+1};t\right)>1-\varepsilon_j$. This implies that $\left\{L_j\right\}_{j\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_j\to L\in X$ as $j\to\infty$. We shall prove that $x\to L\left(S_\lambda^\alpha(I)^{(\mu,\nu)_n}\right)$. For any $\varepsilon>0$ and $t>0$, choose $j\in\mathbb{N}$ such that $\varepsilon_j<\frac{\varepsilon}{4}$,

$\sup_j\nu\left(x_1,x_2,...,x_{n-1},x-x_n^j;t\right)<\frac{\varepsilon}{4}$, $\mu\left(x_1,x_2,...,x_{n-1},L_j-L;t\right)>1-\frac{\varepsilon}{4}$ or $\nu\left(x_1,x_2,...,x_{n-1},L_j-L;t\right)<\frac{\varepsilon}{4}$.

Now since

$$\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}-L;t\right)\ge\varepsilon\right\}\right|$$

$$\le\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}-x_{n_k}^j;\frac{t}{3}\right)\Diamond\left[\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^j-L_j;\frac{t}{3}\right)\Diamond\nu\left(x_1,x_2,...,x_{n-1},L_j-L;\frac{t}{3}\right)\right]\ge\varepsilon\right\}\right|$$

$$\le\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^j-L_j;\frac{t}{3}\right)\ge\frac{\varepsilon}{2}\right\}\right|$$

and similarly

$$\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}-L;t\right)\le1-\varepsilon\right\}\right|>\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^j-L_j;\frac{t}{3}\right)\le1-\frac{\varepsilon}{2}\right\}\right|,$$

it follows that

$$\left\{j\in\mathbb{N}:\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}-L;t\right)\le1-\varepsilon\text{ or }\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}-L;t\right)\ge\varepsilon\right\}\right|\ge\delta\right\}$$

$$\subset\left\{j\in\mathbb{N}:\frac{1}{\lambda_j^\alpha}\left|\left\{k\in I_j:\mu\left(x_1,x_2,...,x_{n-1},x_{n_k}^j-L;\frac{t}{3}\right)\le1-\frac{\varepsilon}{2}\text{ or }\nu\left(x_1,x_2,...,x_{n-1},x_{n_k}^j-L;\frac{t}{3}\right)\ge\frac{\varepsilon}{2}\right\}\right|\ge\delta\right\}$$

for any given $\delta>0$. Hence we have $x\to L\left(S_\lambda^\alpha(I)^{(\mu,\nu)_n}\right)$. So we obtain the proof. □

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