Notes on Intuitionistic Fuzzy Sets
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Differential equation with intuitionistic fuzzy parameters

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Abstract: In this paper, we study differentiability and integrability properties of intuitionistic fuzzy-set-valued mappings and we discuss the existence and uniqueness of solution for differential equations with intuitionistic fuzzy data using the theorem of fixed point in the complete metric space. Then by method of α-cuts we explicit the solution in an example.

Keywords: Differentiability, Integrability, Intuitionistic fuzzy differential equations, Intuitionistic fuzzy solution.
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1 Introduction

In 1965, Zadeh [14] first introduced the fuzzy set theory. Later many researchers have applied this theory to the well known results in the classical set theory. The fuzzy differential equations (FDEs), are originally formulated by Kaleva [8, 9] (see also [13, 10]) and the notions of differential and integral calculus for fuzzy-set-valued are given using Hukuhara difference in Fuzzy theory [10].

Particular in [13], the authors give the existence and uniqueness of the solution of a differential equation with fuzzy initial value, he used the α-cuts. The idea of intuitionistic fuzzy set was first published by Atanassov [1, 2, 3] as a generalization of the notion of fuzzy set.

This paper is organized as follows: in Section 2 we give preliminary which we will use throughout this work. in Section 3 and in Section 4, we give the concept of measurability, integrability and differentiability of an intuitionistic fuzzy valued-functions. Then the existence and uniqueness of the initial value problem for the intuitionistic fuzzy differential equations are treated in Section 5. In Section 6 we present the method of α-cut for explicit the solution with an example in Section 7.

2 Preliminaries

In the first, we recall some results and definitions of intuitionistic fuzzy theory. we denote by

$$\mathrm{IF}_1 = \mathrm{IF}(\mathbb{R}) = \left\{ \langle u, v \rangle \ : \ \mathbb{R} \to [0,1]^2 \,/\, \forall\, x \in \mathbb{R},\ 0 \leq u(x) + v(x) \leq 1 \right\}$$

An element $\langle u, v \rangle$ of IF_1 is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v \rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp\,\langle u, v \rangle = \mathrm{cl}\{x \in \mathbb{R} \ : \mid v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by IF_1
For $\alpha \in [0,1]$ and $\langle u, v \rangle \in \mathrm{IF}_1$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$[\langle u, v \rangle]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u, v \rangle]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}\,.$$

Remark 2.1. *If $\langle u, v \rangle \in IF_1$, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{\langle 1,0 \rangle} \in \mathrm{IF}_1$ as

$$0_{\langle 1,0 \rangle}(t) = \begin{cases} \langle 1, 0 \rangle & t = 0 \\ \langle 0, 1 \rangle & t \neq 0 \end{cases}$$

Let $\langle u, v \rangle$,$\langle u', v' \rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by :

$$\Big(\langle u, v \rangle \oplus \langle u', v' \rangle \Big)(z) = \Big(\sup_{z=x+y} \min\left(u(x), u'(y)\right), \inf_{z=x+y} \max\left(v(x), v'(y)\right) \Big)$$

$$\lambda \langle u, v \rangle = \begin{cases} \langle \lambda u, \lambda v \rangle & \text{if } \lambda \neq 0 \\ 0_{\langle 1,0 \rangle} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u,v\rangle$, $\langle z,w\rangle \in \mathrm{IF}_1$ and $\lambda \in \mathbb{R}$, the addition and scaler-multiplication are defined as follows

$$\begin{aligned}\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]^{\alpha} &= \Big[\langle u,v\rangle\Big]^{\alpha} + \Big[\langle z,w\rangle\Big]^{\alpha},\\ \Big[\lambda\,\langle z,w\rangle\Big]^{\alpha} &= \lambda\Big[\langle z,w\rangle\Big]^{\alpha}\end{aligned} \tag{2.1}$$

$$\begin{aligned}\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]_{\alpha} &= \Big[\langle u,v\rangle\Big]_{\alpha} + \Big[\langle z,w\rangle\Big]_{\alpha},\\ \Big[\lambda\,\langle z,w\rangle\Big]_{\alpha} &= \lambda\Big[\langle z,w\rangle\Big]_{\alpha}\end{aligned} \tag{2.2}$$

Definition 2.1. *Let $\langle u,v\rangle$ an element of IF_1 and $\alpha \in [0,1]$, we define the following sets:*

$$\begin{aligned}\Big[\langle u,v\rangle\Big]_l^{+}(\alpha) &= \inf\{x\in\mathbb{R} \mid u(x)\geq\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^{+}(\alpha) = \sup\{x\in\mathbb{R} \mid u(x)\geq\alpha\}\\ \Big[\langle u,v\rangle\Big]_l^{-}(\alpha) &= \inf\{x\in\mathbb{R} \mid v(x)\leq 1-\alpha\}, \ \Big[\langle u,v\rangle\Big]_r^{-}(\alpha) = \sup\{x\in\mathbb{R} \mid v(x)\leq 1-\alpha\}\end{aligned}$$

Remark 2.2.

$$\Big[\langle u,v\rangle\Big]_{\alpha} = \Big[\Big[\langle u,v\rangle\Big]_l^{+}(\alpha), \Big[\langle u,v\rangle\Big]_r^{+}(\alpha)\Big],\ \Big[\langle u,v\rangle\Big]^{\alpha} = \Big[\Big[\langle u,v\rangle\Big]_l^{-}(\alpha), \Big[\langle u,v\rangle\Big]_r^{-}(\alpha)\Big]$$

Proposition 2.1. *For all α, $\beta \in [0,1]$ and $\langle u,v\rangle \in IF_1$*

(i) $\Big[\langle u,v\rangle\Big]_{\alpha} \subset \Big[\langle u,v\rangle\Big]^{\alpha}$

(ii) $\Big[\langle u,v\rangle\Big]_{\alpha}$ *and* $\Big[\langle u,v\rangle\Big]^{\alpha}$ *are nonempty compact convex sets in* $\mathbb{R}$

(iii) if $\alpha \leq \beta$ then $\Big[\langle u,v\rangle\Big]_{\beta} \subset \Big[\langle u,v\rangle\Big]_{\alpha}$ *and* $\Big[\langle u,v\rangle\Big]^{\beta} \subset \Big[\langle u,v\rangle\Big]^{\alpha}$

(iv) If $\alpha_n \nearrow \alpha$ then $\Big[\langle u,v\rangle\Big]_{\alpha} = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^{\alpha} = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha \in [0,1]$ we denote by

$$M_{\alpha} = \{x\in\mathbb{R} \ : \ u(x)\geq\alpha\} \quad \text{and} \quad M^{\alpha} = \{x\in\mathbb{R} \ : \ v(x)\leq 1-\alpha\}$$

Lemma 2.1. *[11] let $\Big\{M_{\alpha},\ \alpha \in [0,1]\Big\}$ and $\Big\{M^{\alpha},\ \alpha \in [0,1]\Big\}$ two families of subsets of $\mathbb{R}$ satisfies (i)–(iv) in proposition 2.1, if u and v define by*

$$u(x) = \begin{cases} 0 & \text{if } x \notin M_0 \\ \sup\{\alpha\in[0,1] \ : \ x\in M_{\alpha}\} & \text{if } x \in M_0 \end{cases}$$
$$v(x) = \begin{cases} 1 & \text{if } x \notin M^0 \\ 1-\sup\{\alpha\in[0,1] \ : \ x\in M^{\alpha}\} & \text{if } x \in M^0 \end{cases}$$

Then $\langle u,v\rangle \in IF_1$.

Example: A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v:

$$u(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & \text{if } a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2} & \text{if } a_2 \le x \le a_3, \\ 0 & \text{otherwise} \end{cases}$$

$$v(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1^{'}} & \text{if } a_1^{'} \le x \le a_2 \\ \dfrac{x-a_2}{a_3^{'}-a_2} & \text{if } a_2 \le x \le a_3^{'}, \\ 1 & otherwise. \end{cases}$$

where $a_1^{'} \le a_1 \le a_2 \le a_3 \le a_3^{'}$. This TIFN is denoted by $\langle u, v\rangle = \left\langle a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'}\right\rangle$.

$$\begin{aligned} \Big[\langle u, v\rangle\Big]_\alpha &= \Big[a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)\Big] \\ \Big[\langle u, v\rangle\Big]^\alpha &= \Big[a_1' + \alpha(a_2 - a_1'), a_3' - \alpha(a_3' - a_2)\Big] \end{aligned}$$

On the space IF_1 we will consider the following metric,

$$\begin{aligned} d_\infty\Big(\langle u, v\rangle, \langle z, w\rangle\Big) = &\frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u, v\rangle\Big]_r^+(\alpha) - \Big[\langle z, w\rangle\Big]_r^+(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u, v\rangle\Big]_l^+(\alpha) - \Big[\langle z, w\rangle\Big]_l^+(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u, v\rangle\Big]_r^-(\alpha) - \Big[\langle z, w\rangle\Big]_r^-(\alpha)\Big| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\Big|\Big[\langle u, v\rangle\Big]_l^-(\alpha) - \Big[\langle z, w\rangle\Big]_l^-(\alpha)\Big| \end{aligned}$$

Theorem 2.1 ([11]). *$\Big(IF_1, d_\infty\Big)$ is a complete metric space.*

Remark 2.3.

$$\begin{aligned} d_\infty\Big(\langle u, v\rangle, \langle u', v'\rangle\Big) \le &\frac{1}{2}\sup_{0<\alpha\le 1} d_H\Big(\Big[\langle u, v\rangle\Big]_\alpha, \Big[\langle u', v'\rangle\Big]_\alpha\Big) \\ &+ \frac{1}{2}\sup_{0<\alpha\le 1} d_H\Big(\Big[\langle u, v\rangle\Big]^\alpha, \Big[\langle u', v'\rangle\Big]^\alpha\Big) \end{aligned}$$

where d_H is the Hausdorff metric defined in $P_k(\mathbb{R})$ by $d_H([a,b][c,d]) = \max\{|a-c|; |b-d|\}$.

Definition 2.2. *[12]*
Let $\langle u, v\rangle$,$\langle u', v'\rangle \in IF_1$ the H-difference is the IFN $\langle z, w\rangle \in IF_1$, if it exists, such that

$$\langle u, v\rangle \ominus \langle u', v'\rangle = \langle z, w\rangle \Longleftrightarrow \langle u, v\rangle = \langle u', v'\rangle \oplus \langle z, w\rangle$$

Definition 2.3. *Let $F : IF_1 \to IF_1$ be an intuitionistic fuzzy valued mapping and $\langle u, v\rangle \in IF_1$. Then F is called intuitionistic fuzzy continuous in $\langle u, v\rangle$ iff:*

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall \langle z, w\rangle \in IF_1)\Big(d_\infty\Big(\langle u, v\rangle , \langle z, w\rangle\Big) < \delta\Big) \Rightarrow d_\infty\Big(F(\langle u, v\rangle), F(\langle z, w\rangle)\Big) < \varepsilon$$

Definition 2.4. *Let $F : [a, b] \to IF_1$ be an intuitionistic fuzzy valued mapping and $t_0 \in [a, b]$. Then F is called intuitionistic fuzzy continuous in t_0 iff:*

$$(\forall\varepsilon > 0)(\exists\delta > 0)\Big(\forall t \in [a, b] \textit{such as} \mid t - t_0 \mid < \delta\Big) \Rightarrow d_\infty(F(t), F(t_0)) < \varepsilon$$

Definition 2.5. *F is called intuitionistic fuzzy continuous iff is intuitionistic fuzzy continuous in every point of $[a, b]$.*

Definition 2.6. *A mapping $F : [a, b] \to IF_1$ is said to be differentiable at $t_0 \in (a, b)$ if there exist $F'(t_0) \in IF_1$ such that both limits:*

$$\lim_{\Delta t\to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t} \quad \textit{and} \quad \lim_{\Delta t\to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t} \tag{2.3}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called intuitionistic fuzzy derivative of F at t_0. Here the limit is taken in the metric space (IF_1, d_∞). At the end points of $[a, b]$ we consider only the one-sided derivatives.

Definition 2.7. *Let $F : [a, b] \to IF_1$ be an intuitionistic fuzzy valued mapping. Let $P : [a, b] \to IF_1$ be differentiable mappings at every $t \in (a, b)$. P is said to be a primitive of F if the intuitionistic fuzzy derivative of P equals F for every $t \in (a, b)$, that is, $P'(t) = F(t)$.*

3 Measurability

Throughout this paper, $(\mathbb{R}, B(\mathbb{R}), \mu)$ denotes a complete finite measure space.
Let us $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

$F : [a, b] \to \mathrm{IF}_1$ is called integrably bounded if there exists an integrable function h such that $|y| \le h(t)$ holds for any $y \in supp(F(t)), t \in [a, b]$.

Definition 3.1. *we say that a mapping $F : [a, b] \to IF_1$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued mapping $F_\alpha : [a, b] \to P_k(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]_\alpha$ and $F^\alpha : [a, b] \to P_k(\mathbb{R})$ defined by $F^\alpha(t) = [F(t)]^\alpha$ are (Lebesgue) measurable, when $P_k(\mathbb{R})$ is endowed with the topology generated the Hausdorff metric d_H*

Lemma 3.1. *If $F : [a, b] \to IF_1$ is continuous then it is strongly measurable.*

Proof. Let $\varepsilon > 0$ and $t_0 \in [a, b]$, by continuity there exists a $\delta > 0$ such that

$$d_\infty(F(t), F(t_0)) < \varepsilon \text{ whenever } \mid t - t_0 \mid < \delta$$

$d_\infty(F(t), F(t_0)) < \varepsilon \Rightarrow$

$$|[(F(t))]_r^+(\alpha)-[F(t_0)]_r^+(\alpha)|<\varepsilon \text{ and } |[F(t)]_l^+(\alpha)-[F(t_0)]_l^+(\alpha)|<\varepsilon$$

hence
$\max\left(|[(F(t))]_r^+(\alpha)-[F(t_0)]_r^+(\alpha)|;|[F(t)]_l^+(\alpha)-[F(t_0)]_l^+(\alpha)|\right)=d_H\Big(F_\alpha(t),F_\alpha(t_0)\Big)<\varepsilon$
whenever $\mid t-t_0\mid<\delta$. So F_α is continuous with respect to the Hausdorff metric.
Therefore $F_\alpha^{-1}(U)$ is open, for each open U in $P_k(\mathbb{R})$.In the same way we show that F^α is measurable. □

Lemma 3.2. *Let $F:[a,b]\to IF_1$ be strongly measurable and denote $F_\alpha(t)=\Big[\lambda_\alpha(t),\lambda^\alpha(t)\Big]$, $F^\alpha(t)=\Big[\mu_\alpha(t),\mu^\alpha(t)\Big]$ for $\alpha\in[0,1]$. Then $\lambda_\alpha,\lambda^\alpha,\mu_\alpha,\mu^\alpha$ are measurable.*

Proof. Let $\alpha\in[0,1]$ be fixed. Then F_α and F^α are measurable and closed valued.
Consequently its have a Castaing representation ([6])i.e., there exists a sequence g_k^α of measurable selections such that for all $t\in[a,b]$,

$$F_\alpha(t)=\overline{\{g_k^\alpha,|k=1,2,...\}}$$

But from the definition of $F_\alpha(t)$, it follows that λ_α = inf g_k^α and λ^α =sup g_k^α. In the same way we show that μ^α and μ_α are measurable, which proves the lemma. □

4 Integrability

Definition 4.1. *Suppose $A=[a,b]$, $F:A\to IF_1$ is integrably bounded and strongly measurable for each $\alpha\in(0,1]$ write*

$$\left[\int_A F(t)dt\right]_\alpha=\int_A[F(t)]_\alpha\,dt=\left\{\int_A fdt|f:A\to\mathbb{R}\textit{ is a measurable selection for }F_\alpha\right\}.$$

$$\left[\int_A F(t)dt\right]^\alpha=\int_A[F(t)]^\alpha\,dt=\left\{\int_A fdt|f:A\to\mathbb{R}\textit{ is a measurable selection for }F^\alpha\right\}.$$

if there exists $\langle u,v\rangle\in IF_1$ such that $\Big[\langle u,v\rangle\Big]^\alpha=\Big[\int_A F(t)dt\Big]^\alpha$ and $\Big[\langle u,v\rangle\Big]_\alpha=\Big[\int_A F(t)dt\Big]_\alpha$ $\forall\alpha\in(0,1]$. Then F is called integrable on A, write $\langle u,v\rangle=\int_A F(t)dt$.

Remark 4.1.
- *If $F(t)=\langle u_t,v_t\rangle$ is integrable,then $\int\langle u_t,v_t\rangle=\left\langle\int u_t,\int v_t\right\rangle$,*
- *If $F:[a,b]\to IF_1$ is integrable then in view of Lemma (3.2) $\int F$ is obtained by integrating the α-level curves, that is*

$$\left[\int F\right]_\alpha=\left[\int\lambda_\alpha,\int\lambda^\alpha\right]\textit{ and }\left[\int F\right]^\alpha=\left[\int\mu_\alpha,\int\mu^\alpha\right],\alpha\in[0,1]$$

$$F_\alpha(t)=[F(t)]_\alpha=\Big[\lambda_\alpha(t),\lambda^\alpha(t)\Big],F^\alpha(t)=[F(t)]^\alpha=\Big[\mu_\alpha(t),\mu^\alpha(t)\Big]\textit{ for }\alpha\in[0,1].$$

Theorem 4.1. *If $F:[a,b]\to IF_1$ is strongly measurable and integrably bounded, then F is integrable.*

Proof. We denote $\mathcal{M}_\alpha = \int F_\alpha$ and $\mathcal{M}^\alpha = \int F^\alpha$, Since F is strongly measurable and integrably bounded, then due to [4] there exist two fuzzy numbers u and $1-v$ such that

$$\mathcal{M}_\alpha = \{x \in \mathbb{R} \,:\, u(x) \geq \alpha\} \quad \text{and} \quad \mathcal{M}^\alpha = \{x \in \mathbb{R} \,:\, 1-v(x) \geq \alpha\}$$

Then properties (i)-(iii) of lemma (2.1) are checked.
Since $F_\alpha \subset F^\alpha \Rightarrow \int F_\alpha \subset \int F^\alpha$ for all $\alpha \in [0,1]$, by Lemma (2.1), There exists unique $\langle u,v\rangle \in \mathrm{IF}_1$ such that $\Big[\langle u,v\rangle\Big]^\alpha = \int F^\alpha$ et $\Big[\langle u,v\rangle\Big]_\alpha = \int F_\alpha$, which completes the proof. □

Corollary 4.1. *If $F : [a,b] \to IF_1$ is continuous, then it is integrable.*

Proof. F is continuous $\Rightarrow$ F is strongly measurable.
Since F^0 is continuous, $F^0(t) \in P_k(\mathbb{R})$ for all $t \in [a,b]$ and $[a,b]$ is compact, then $\cup_{t\in[a,b]}F^0(t)$ is compact. So F is integrably bounded, which completes the proof. □

Theorem 4.2. *Let $F : [a,b] \to IF_1$ be integrable and $c \in [a,b]$. Then*

$$\int_a^b F = \int_a^c F \oplus \int_c^b F.$$

Proof. Let $\alpha \in [0,1]$ and f be a measurable selection for F_α.
Since $\int_a^b f = \int_a^c f + \int_c^b f$, then we get

$$\left[\int_a^b F\right]_\alpha \subset \left[\int_a^c F\right]_\alpha + \left[\int_c^b F\right]_\alpha.$$

On the other hand, let $z = \int_a^c g_1 + \int_c^b g_2$ where g_1 is a measurable selection for F_α in $[a,c]$ and g_2 is a measurable selection for F_α in $[c,b]$. Then f defined by

$$f(t) = \begin{cases} g_1(t) & if \;\; t \in [a,c] \\ g_2(t) & if \;\; t \in [c,b]. \end{cases}$$

is measurable selection F_α in $[a,b]$ and $\int_a^b f = \int_a^c g_1 + \int_c^b g_2 = z$ hence

$$\left[\int_a^c F\right]_\alpha + \left[\int_c^b F\right]_\alpha \subset \left[\int_a^b F\right]_\alpha.$$

in the same way we show that

$$\left[\int_a^c F\right]^\alpha + \left[\int_c^b F\right]^\alpha = \left[\int_a^b F\right]^\alpha.$$

□

Theorem 4.3. *Let F, G :$[a,b] \to IF_1$ be integrable and $\lambda \in \mathbb{R}$. Then*

1. $\int (F(t) \oplus G(t))dt = \int F(t) \oplus \int G(t)$,
2. $\int (\lambda F(t))dt = \lambda \int F(t)dt$,

3. $d_\infty\Big(F(t), G(t)\Big)$ *is integrable,*

4. $d_\infty\left(\int F(t)dt, \int G(t)dt\right) \leq \int d_\infty\left(F(t), G(t)\right)dt$

Proof. Let $\alpha \in [0,1]$, the upper and lower α-cuts of $F(t)$ and $G(t)$ respectively given by $F_\alpha(t), F^\alpha(t), G_\alpha(t), G^\alpha(t)$ are compact-convex-valued. Since the space $P_k(\mathbb{R})$ can be embedded into a Banach space and it follows from Debreu[5] that the integrals $\int F_\alpha(t)$, $\int F^\alpha(t)$,$\int G_\alpha(t)$ and $\int G^\alpha(t)$ are in fact Bochner integrals. Hence applying Eqs. (2.1), (2.2) we obtain

$$\int [F(t) \oplus G(t)]_\alpha = \int F_\alpha(t) + \int G_\alpha(t)$$

$$\int [F(t) \oplus G(t)]^\alpha = \int F^\alpha(t) + \int G^\alpha(t)$$

which proves 1.
A similar reasoning yields 2.
Now for 3. using Remark 2.3 we have

$$d_\infty\Big(F(t), G(t)\Big) \leq \frac{1}{2}\sup_{0<\alpha\leq 1} d_H\Big(\Big[F(t)\Big]_\alpha, \Big[G(t)\Big]_\alpha\Big) + \frac{1}{2}\sup_{0<\alpha\leq 1} d_H\Big(\Big[F(t)\Big]^\alpha, \Big[G(t)\Big]^\alpha\Big)$$

Since, $\sup\limits_{0<\alpha\leq 1} d_H\Big(\Big[F(t)\Big]_\alpha, \Big[G(t)\Big]_\alpha\Big)$ and $\sup\limits_{0<\alpha\leq 1} d_H\Big(\Big[F(t)\Big]^\alpha, \Big[G(t)\Big]^\alpha\Big)$ are integrable see [Theorem 4.3,(iii) [8]]. Hence,$d_\infty\Big(F(t), G(t)\Big)$ is integrable.
Finally, by the definition of metric d_∞ we have

$$\begin{aligned}
d_\infty & \Big(\int F(t), \int G(t)\Big) = \\
= & \ \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|\int [(F(t))]_l^+(\alpha) - \int [G(t)]_l^+(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|\int [(F(t))]_r^+(\alpha) - \int [G(t)]_r^+(\alpha)\Big| \\
+ & \ \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|\int [(F(t))]_l^-(\alpha) - \int [G(t)]_l^-(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\Big|\int [(F(t))]_r^-(\alpha) - \int [G(t)]_r^-(\alpha)\Big| \\
\leq & \ \frac{1}{4}\sup_{0<\alpha\leq 1}\int\Big|[(F(t))]_l^+(\alpha) - [G(t)]_l^+(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\int\Big|[(F(t))]_r^+(\alpha) - [G(t)]_r^+(\alpha)\Big| \\
& + \frac{1}{4}\sup_{0<\alpha\leq 1}\int\Big|[(F(t))]_l^-(\alpha) - [G(t)]_l^-(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\leq 1}\int\Big|[(F(t))]_r^-(\alpha) - [G(t)]_r^-(\alpha)\Big| \\
\leq & \ \int\frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(F(t))]_l^+(\alpha) - [G(t)]_l^+(\alpha)\Big| + \int\frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(F(t))]_r^+(\alpha) - [G(t)]_r^+(\alpha)\Big| \\
& + \int\frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(F(t))]_l^-(\alpha) - [G(t)]_l^-(\alpha)\Big| + \int\frac{1}{4}\sup_{0<\alpha\leq 1}\Big|[(F(t))]_r^-(\alpha) - [G(t)]_r^-(\alpha)\Big|
\end{aligned}$$

Thus,

$$d_\infty\left(\int F(t), \int G(t)\right) \leq \int d_\infty\left(F(t), G(t)\right)$$

$\square$

Lemma 4.1. *Let $A \in IF_1$ and $F : [a,b] \to IF_1$ by $F(s) = A$ for all $s \in [a,b]$. Then $\int_a^b F = (b-a)A$*

Proof.

$$\begin{aligned}
\left[\int_a^b F(s)ds\right]_\alpha &= \int_a^b [F(s)]_\alpha ds = \int_a^b [A]_\alpha ds = (b-a)[A]_\alpha \\
\left[\int_a^b F(s)ds\right]^\alpha &= \int_a^b [F(s)]^\alpha ds = \int_a^b [A]^\alpha ds = (b-a)[A]^\alpha
\end{aligned}$$

So

$$\int_a^b F(s)ds = (b-a)A$$

□

Theorem 4.4. *If $F : [a,b] \to IF_1$ is continuous, then $\int_a^t F$ is Lipschitz continuous on $[a,b]$.*

Proof. Let $s, t \in [a,b]$ and assume that $t < s$. Then

$$\begin{aligned}
d_\infty\left(\int_a^s F(\tau)d\tau, \int_a^t F(\tau)d\tau\right) &= d_\infty\left(\int_a^t F(\tau)d\tau + \int_t^s F(\tau)d\tau, \int_a^t F(\tau)d\tau\right) \\
&\leq d_\infty\left(\int_t^s F(\tau)d\tau, 0_{\langle 1,0\rangle}(\tau)\right) \\
&\leq \int_t^s d_\infty\Big(F(\tau), 0_{\langle 1,0\rangle}(\tau)\Big)d\tau
\end{aligned}$$

Since $\cup_{t\in[a,b]} F^0(t)$ is compact, then there exists an $M > 0$ such that $|x| \leq M$ for all $x \in F^0(t)$ and $t \in [a,b]$, this implies that $d_\infty\Big(F(\tau), 0_{\langle 1,0\rangle}(\tau)\Big) \leq M$, by Lemma4.1

Then we have

$$d_\infty\left(\int_a^s F, \int_a^t F\right) \leq M(s-t)$$

□

Theorem 4.5. *Let $F : [a,b] \to IF_1$ be continuous. Then for all $t \in [a,b]$ the integral $G(t) = \int_a^t F$ is differentiable and $G'(t) = F(t)$.*

Proof. F is continuous $\Rightarrow$ F is integrable.
Let $\varepsilon > 0$, for $h > 0$ $G(t+h) \ominus G(t) = \int_t^{t+h} F$ then,

$$\begin{aligned}
d_\infty\left(\frac{1}{h}(G(t+h) \ominus G(t)), F(t)\right) &= \frac{1}{h} d_\infty\left(\int_t^{t+h} F(s)ds, hF(t)\right) \\
&= \frac{1}{h} d_\infty\left(\int_t^{t+h} F(s)ds, \int_t^{t+h} F(t)ds\right) \\
&\leq \frac{1}{h}\int_t^{t+h} d_\infty\Big(F(s), F(t)\Big)
\end{aligned}$$

Since F is continuous, then $d_\infty\left(\frac{1}{h}(G(t+h)\ominus G(t)), F(t)\right) < \varepsilon$
hence $\lim_{h\to 0}(G(t+h)\ominus G(t))/h = F(t)$, and similarly $\lim_{h\to 0}\frac{1}{h}(G(t)\ominus G(t-h)) = F(t)$, which proves the theorem. $\square$

Theorem 4.6. *[7] Let $F : T \to IF_1$ be differentiable and assume that the derivative F' is integrable over T. Then, for each $s \in T$, we have*

$$F(s) = F(a) \oplus \int_a^s F'(t)dt.$$

Theorem 4.7. *Let $F : [a,b] \to IF_1$ be differentiable. Denote $F^\alpha(t) = [F(t)]^\alpha = [\lambda_\alpha(t), \lambda^\alpha(t)]$, $F_\alpha(t) = [F(t)]_\alpha = [\mu_\alpha(t), \mu^\alpha(t)]$. Then $\lambda_\alpha(t), \lambda^\alpha(t), \mu_\alpha(t)$ and $\mu^\alpha(t)$ are differentiable and $[F(t)']^\alpha = [\lambda'_\alpha(t), \lambda^{\alpha\prime}(t)]$, $[F(t)']_\alpha = [\mu'_\alpha(t), \mu^{\alpha\prime}(t)]$*

Proof. we prove that for F^α,and its similarly for F_α.
Now $\Big[F(t+h)\ominus F(t)\Big]^\alpha = \Big[\lambda_\alpha(t+h)-\lambda_\alpha(t), \lambda^\alpha(t+h)-\lambda^\alpha(t)\Big]$,
and $\Big[F(t)\ominus F(t-h)\Big]^\alpha = \Big[\lambda_\alpha(t)-\lambda_\alpha(t-h), \lambda^\alpha(t)-\lambda^\alpha(t-h)\Big]$.
Dividing by h and passing to the limit gives the conclusion. $\square$

Proposition 4.1. *Let $F : [a,b] \to IF_1$ and $G : [a,b] \to IF_1$ be differentiable mappings. If F and G are both primitives of the same mapping and there exists $F(t)\ominus G(t)$ for every $t \in (a,b)$, then $F(t) = G(t)\oplus C$, being $C \in IF_1$.*

Proof. Let $F(t) = G(t)\oplus C(t)$, By taking the intuitionistic fuzzy derivative at both sides, we have that $F'(t) = G'(t)\oplus C'(t)$, and hence $C'(t) = 0_{\langle 1,0\rangle}$ for every $t \in (a,b)$ which implies that C is constant. $\square$

Theorem 4.8. *If $F : [a,b] \to IF_1$ is differentiable then it is continuous.*

Proof. Let $t, t+h \in [a,b]$ with $h > 0$.

$$\begin{aligned} d_\infty\Big(F(t+h), F(t)\Big) &= d_\infty\Big(F(t+h)\ominus F(t), 0_{\langle 1,0\rangle}\Big) \\ &\leq hd_\infty\Big(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\Big) + hd_\infty(F'(t), 0_{\langle 1,0\rangle}) \end{aligned}$$

where h is so small that the H-difference $F(t+h)\ominus F(t)$ exists.
When $h \to 0$ the right-hand side goes to 0 and hence F is right continuous. The left continuity is similarly proven. $\square$

5 Existence and uniqueness

In this section we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \langle u_{t_0}(.), v_{t_0}(.)\rangle \end{cases} \tag{5.1}$$

where $x \in \mathrm{IF}_1$ is unknown, $I = [t_0, T]$ and $f : I \times \mathrm{IF}_1 \to \mathrm{IF}_1$.
Denote by $C(I, \mathrm{IF}_1)$ the set of all continuous mappings from I to IF_1.
Defining the metric

$$D\,(f, g) = \sup_{t\in I} d_\infty\Big(f(t), g(t)\Big)$$

with $f(t) = \langle f_{1,t}, f_{2,t}\rangle$ et $g(t) = \langle g_{1,t}, g_{2,t}\rangle$.

Theorem 5.1. $(C(I, IF_1), D)$ *is a complete metric space.*

Proof. Let $(f_n)_n$ be a sequence of Cauchy in $(C(I, \mathrm{IF}_1), D)$ then, $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall n, q \in \mathbb{N}$, $n, q \geq N \Rightarrow D(f_n, f_q) \leq \varepsilon$
i.e $\forall n, q \in \mathbb{N}$, $n, q \geq N \Rightarrow \sup_{t\in I} d_\infty(f_n(t), f_q(t)) \leq \varepsilon$
which implies that $\exists N \in \mathbb{N}$, $\forall n, q \in \mathbb{N}$, $\forall t \in I$, $n, q \geq N \Rightarrow d_\infty(f_n(t), f_q(t)) \leq \varepsilon$.
Since $(\mathrm{IF}_1, d_\infty)$ is a complete metric space then, there exists $f(t) \in \mathrm{IF}_1$, $\forall t \in I$ such that $d_\infty(f_n(t), f(t)) \to 0$ as $n \to +\infty$. Thus, $D(f_n, f) \to 0$ as $n \to +\infty$.
It remains to prove that $f \in C(I, \mathrm{IF}_1)$, so $\forall n \in \mathbb{N}$, $\forall t, t_0 \in I$; we have

$$\begin{aligned} d_\infty(f(t), f(t_0)) &\leq d_\infty(f(t), f_n(t)) + d_\infty(f_n(t), f_n(t_0)) + d_\infty(f_n(t_0), f(t_0)) \\ &\leq D(f, f_n) + d_\infty(f_n(t), f_n(t_0)) + d_\infty(f_n(t_0), f(t_0)) \end{aligned}$$

for sufficiently large n and t belongs to the neighborhood of t_0. So $f \in C(I, \mathrm{IF}_1)$. □

Definition 5.1. $x : I \to IF_1$ *is a solution of the initial value problem* (5.1)*, if and only if it is continuous and satisfies the integral equation*

$$x(t) = x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds, \quad \textit{for all } \ t \in I.$$

Denote by $C(I \times \mathrm{IF}_1, \mathrm{IF}_1)$ the set of all continuous mappings from $I \times \mathrm{IF}_1$ to IF_1

Theorem 5.2. *Assume that* $f \in C(I \times IF_1, IF_1)$ *and there exists a constant* $k > 0$ *such that*

$$\begin{aligned} |[f(s, x(s))]_r^+(\alpha) - [f(s, y(s))]_r^+(\alpha)| &\leq k|[x(s)]_r^+(\alpha) - [y(s)]_r^+(\alpha)| \\ |[f(s, x(s))]_l^+(\alpha) - [f(s, y(s))]_l^+(\alpha)| &\leq k|[x(s)]_l^+(\alpha) - [y(s)]_l^+(\alpha)| \\ |[f(s, x(s))]_r^-(\alpha) - [f(s, y(s))]_r^-(\alpha)| &\leq k|[x(s)]_r^-(\alpha) - [y(s)]_r^-(\alpha)| \\ |[f(s, x(s))]_l^-(\alpha) - [f(s, y(s))]_l^-(\alpha)| &\leq k|[x(s)]_l^-(\alpha) - [y(s)]_l^-(\alpha)| \end{aligned}$$

with $k(T - t_0) < 1$*, for all* $s \in I, x, y \in IF_1$*. Then the problem (5.1) has an unique solution on* I*.*

Proof. For $x \in C(I, \mathrm{IF}_1)$, we define Px on I by the relation

$$Px(t) = x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds$$

we put $\varphi(t) = Px(t+h)$ and $\psi(t) = Px(t)$
we have

$$\begin{aligned}
D(\varphi, \psi) &= D\Big(x(t_0) \oplus \int_{t_0}^{t+h} f(s, x(s))ds, x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds\Big) \\
&= D\Big(\int_{t_0}^{t+h} f(s, x(s))ds, \int_{t_0}^{t} f(s, x(s))ds\Big) \\
&= \sup_{t \in I} d_\infty\Big(\int_{t_0}^{t+h} f(s, x(s))ds, \int_{t_0}^{t} f(s, x(s))ds\Big)
\end{aligned}$$

Then

$$\begin{aligned}
D(\varphi, \psi) = \sup_{t \in I} \Big\{ &\frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t_0}^{t+h} [f(s, x(s))]_r^+(\alpha)ds - \int_{t_0}^{t} [f(s, x(s))]_r^+(\alpha)ds \Big| \\
&+ \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t_0}^{t+h} [f(s, x(s))]_l^+(\alpha)ds - \int_{t_0}^{t} [f(s, x(s))]_l^+(\alpha)ds \Big| \\
&+ \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t_0}^{t+h} [f(s, x(s))]_r^-(\alpha)ds - \int_{t_0}^{t} [f(s, x(s))]_r^-(\alpha)ds \Big| \\
&+ \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t_0}^{t+h} [f(s, x(s))]_l^-(\alpha)ds - \int_{t_0}^{t} [f(s, x(s))]_l^-(\alpha)ds \Big| \Big\}
\end{aligned}$$

Then

$$\begin{aligned}
D(\varphi, \psi) = \sup_{t \in I} \Big\{ &\frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t}^{t+h} [f(s, x(s))]_r^+(\alpha)ds \Big| + \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t}^{t+h} [f(s, x(s))]_l^+(\alpha)ds \Big| \\
&+ \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t}^{t+h} [f(s, x(s))]_r^-(\alpha)ds \Big| + \frac{1}{4} \sup_{0<\alpha\leq 1} \Big| \int_{t}^{t+h} [f(s, x(s))]_l^-(\alpha)ds \Big| \Big\}
\end{aligned}$$

when $h \to 0$, $D(\varphi, \psi) \to 0$. Therefore $Px \in C(I, \mathrm{IF}_1)$.
Now, let $x, y \in C(I, \mathrm{IF}_1)$ we have

$$\begin{aligned}
D(Px, Py) &= D\Big(x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds, x(t_0) \oplus \int_{t_0}^{t} f(s, y(s))ds\Big) \\
&= D\Big(\int_{t_0}^{t} f(s, x(s))ds, \int_{t_0}^{t} f(s, y(s))ds\Big) \\
&= \sup_{t \in I} d_\infty\Big(\int_{t_0}^{t} f(s, x(s))ds, \int_{t_0}^{t} f(s, y(s))ds\Big) \\
&\leq \sup_{t \in I} \int_{t_0}^{t} d_\infty\Big(f(s, x(s)), f(s, y(s))\Big)ds
\end{aligned}$$

We get

$$D(Px,Py) \le \sup_{t\in I}\int_{t_0}^{t}\left\{\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_r^+(\alpha)-[f(s,y(s))]_r^+(\alpha)\Big|\right.$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_l^+(\alpha)-[f(s,y(s))]_l^+(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_r^-(\alpha)-[f(s,y(s))]_r^-(\alpha)\Big|$$
$$\left.+\,\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_l^-(\alpha)-[f(s,y(s))]_l^-(\alpha)\Big|\right\}ds$$

Then

$$D(Px,Py) \le \sup_{t\in I}(t-t_0)\left\{\frac{1}{4}\sup_{0<\alpha\le 1}\sup_{t_0\le s\le t}\Big|[f(s,x(s))]_r^+(\alpha)-[f(s,y(s))]_r^+(\alpha)\Big|\right.$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\sup_{t_0\le s\le t}\Big|[f(s,x(s))]_l^+(\alpha)-[f(s,y(s))]_l^+(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\sup_{t_0\le s\le t}\Big|[f(s,x(s))]_r^-(\alpha)-[f(s,y(s))]_r^-(\alpha)\Big|$$
$$\left.+\,\frac{1}{4}\sup_{0<\alpha\le 1}\sup_{t_0\le s\le t}\Big|[f(s,x(s))]_l^-(\alpha)-[f(s,y(s))]_l^-(\alpha)\Big|\right\}$$

$$D(Px,Py) \le (T-t_0)\sup_{s\in I}\left\{\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_r^+(\alpha)-[f(s,y(s))]_r^+(\alpha)\Big|\right.$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_l^+(\alpha)-[f(s,y(s))]_l^+(\alpha)\Big|$$
$$+\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_r^-(\alpha)-[f(s,y(s))]_r^-(\alpha)\Big|$$
$$\left.+\,\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[f(s,x(s))]_l^-(\alpha)-[f(s,y(s))]_l^-(\alpha)\Big|\right\}$$

$$D(Px,Py) \le$$
$$\le k(T-t_0)\sup_{s\in I}\left\{\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[x(s)]_r^+(\alpha)-[y(s)]_r^+(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\le 1}\Big|[x(s)]_l^+(\alpha)-[y(s)]_l^+(\alpha)\Big|\right.$$
$$\left.+\,\frac{1}{4}\sup_{0<\alpha\le 1}\Big|[x(s)]_r^-(\alpha)-[y(s)]_r^-(\alpha)\Big| + \frac{1}{4}\sup_{0<\alpha\le 1}\Big|[x(s)]_l^-(\alpha)-[y(s)]_l^-(\alpha)\Big|\right\}$$

Therefore $D(Px,Py) \le k(T-t_0)D(x,y)$.
The contraction mapping principle assures that there exists an unique fixed point of P. $\square$

6 Solving intuitionistic fuzzy differential equations

In this section we give a procedure to solve the intuitionistic fuzzy differential equations

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \langle u_{t_0}(.), v_{t_0}(.)\rangle \end{cases} \tag{6.1}$$

Denote

$$\Big[x(t)\Big]_\alpha = \Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big]$$

$$\Big[x(t)\Big]^\alpha = \Big[[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big]$$

$$\Big[x(t_0)\Big]_\alpha = \Big[[x(t_0)]_l^+(\alpha), [x(t_0)]_r^+(\alpha)\Big]$$

$$\Big[x(t_0)\Big]^\alpha = \Big[[x(t_0)]_l^-(\alpha), [x(t_0)]_r^-(\alpha)\Big]$$

$$\Big[f(t, x(t))\Big]_\alpha = \Big[g\Big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big),\ h\Big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big)\Big]$$

$$\Big[f(t, x(t))\Big]^\alpha = \Big[l\Big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big),\ k\Big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big)\Big]$$

and proceeded as follows

1. solve the system
$$\begin{cases} [x^{'}(t)]_l^+(\alpha) = g\Big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big) & ;\ \ [x(t_0)]_l^+(\alpha) \\ [x^{'}(t)]_r^+(\alpha) = h\Big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big) & ;\ \ [x(t_0)]_r^+(\alpha) \\ [x^{'}(t)]_l^-(\alpha) = l\Big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big) & ;\ \ [x(t_0)]_l^-(\alpha) \\ [x^{'}(t)]_r^-(\alpha) = k\Big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big) & ;\ \ [x(t_0)]_r^-(\alpha) \end{cases}$$

2. Denote $\Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big] = M_\alpha$, $\Big[[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big] = M^\alpha$
and
$\Big[[x'(t)]_l^+(\alpha), [x'(t)]_r^+(\alpha)\Big] = M'_\alpha$, $\Big[[x'(t)]_l^-(\alpha), [x'(t)]_r^-(\alpha)\Big] = M'^\alpha$.
ensure that $\big(\ M_\alpha, M^\alpha\ \big)$ and $\big(M'_\alpha, M'^\alpha\ \big)$ verifying $(i) - (iv)$ of lemma (2.1).

3. using lemma (2.1), there exists $\langle u, v\rangle \in \mathrm{IF}_1$ such that $\langle u, v\rangle = x(t)$.

7 Example

Consider the problem

$$\begin{cases} x^{'}(t) + x(t) = \sigma(t) \quad ,\ t \geq 0 \\ \qquad\qquad x(0) = x_0 \end{cases} \tag{7.1}$$

where $x_0 = \langle -1, 0, 1; -\frac{3}{2}, 0, \frac{3}{2} \rangle$ and $\sigma(t) = 2\exp(-t)x_0$

If the conditions of Theorem 5.2 are satisfied, then the problem has a unique solution.

We get the differential system

$$\begin{cases} [x^{'}(t)]_l^+(\alpha) + [x(t)]_l^+(\alpha) = 2(\alpha - 1)\exp(-t) & , \ [x(0)]_l^+(\alpha) = \alpha - 1 \\ [x^{'}(t)]_r^+(\alpha) + [x(t)]_r^+(\alpha) = 2(1 - \alpha)\exp(-t) & , \ [x(0)]_r^+(\alpha) = 1 - \alpha \\ [x^{'}(t)]_l^-(\alpha) + [x(t)]_l^-(\alpha) = 3(\alpha - 1)\exp(-t) & , \ [x(0)]_l^-(\alpha) = \frac{3}{2}(\alpha - 1) \\ [x^{'}(t)]_r^-(\alpha) + [x(t)]_+^-(\alpha) = 3(1 - \alpha)\exp(-t) & , \ [x(0)]_+^-(\alpha) = \frac{3}{2}(1 - \alpha) \end{cases}$$

we get

$$[x(t)]_l^+(\alpha) = (\alpha - 1)\exp(-t)(1 + 2t)$$
$$[x(t)]_r^+(\alpha) = (1 - \alpha)\exp(-t)(1 + 2t)$$
$$[x(t)]_l^-(\alpha) = (3t + \frac{3}{2})(\alpha - 1)\exp(-t)$$
$$[x(t)]_r^-(\alpha) = (3t + \frac{3}{2})(1 - \alpha)\exp(-t)$$

Therfore

$$\Big[x(t)\Big]_\alpha = \Big[(\alpha - 1)(1 + 2t)\exp(-t), (1 - \alpha)(1 + 2t)\exp(-t)\Big]$$
$$\Big[x(t)\Big]^\alpha = \Big[(\alpha - 1)(3t + \frac{3}{2})\exp(-t), (1 - \alpha)(3t + \frac{3}{2})\exp(-t)]$$

We see that $[x(t)]_l^+(\alpha) \leq [x(t)]_r^+(\alpha)$ and $[x(t)]_l^-(\alpha) \leq [x(t)]_r^-(\alpha)$ only if $t \geq -\frac{1}{2}$
also,

$$[x^{'}(t)]_l^+(\alpha) = (\alpha - 1)(1 - 2t)\exp(-t) \leq [x^{'}(t)]_r^+(\alpha) = (1 - \alpha)(1 - 2t)\exp(-t)$$

and

$$[x^{'}(t)]_l^-(\alpha) = (\alpha - 1)(\frac{3}{2} - 3t)\exp(-t) \leq [x^{'}(t)]_r^-(\alpha) = (1 - \alpha)(\frac{3}{2} - 3t)\exp(-t)$$

only if $t \leq \frac{1}{2}$.
Moreover, we observe that $\Big[x(t)\Big]_\alpha \subseteq \Big[x(t)\Big]^\alpha$ for all $\alpha \in [0, 1]$.
So, $x(t)$ define an unique intuitionistic fuzzy solution to the problem (7.1) on the interval $\Big[0, \frac{1}{2}\Big]$.

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